CLASS XII - CBSE

MATHEMATICS (SOLUTION)

SECTION – A

1. Given $f'(x) = x^2 - \frac{1}{x^2}$,

$$f(x) = \int\left(x^2 - \frac{1}{x^2}\right)dx \qquad \text{[By definition]}$$

$$= \int (x^2 - x^{-2})dx$$

$$= \frac{x^3}{3} - \frac{x^{-1}}{-1} + C = \frac{1}{3}x^3 + \frac{1}{x} + C$$ **[1]**

Also $f(1) = \frac{1}{3}$ $\Rightarrow \frac{1}{3} \times 1^3 + \frac{1}{1} + C = \frac{1}{3}$

$$\Rightarrow \frac{1}{3} + 1 + C = \frac{1}{3} \Rightarrow C = -1$$

$$\therefore f(x) = \frac{1}{3}x^3 + \frac{1}{x} - 1.$$ **[1]**

OR

Let $I = \int \frac{\log x}{(1+\log x)^2}dx$

Putting $\log x = t$;

$\Rightarrow x = e^t$ and $dx = e^t\,dt$ **[½]**

$$\therefore I = \int \frac{t}{(1+t)^2}.e^t\,dt$$

$$= \int e^t.\frac{(1+t-1)}{(1+t)^2}dt$$

$$= \int e^t.\left\{\frac{1}{(1+t)} - \frac{1}{(1+t)^2}\right\}dt$$ **[½]**

$$= \frac{e^t}{(1+t)} + C = \frac{x}{(1+\log x)} + C$$ **[1]**

2. Given differential eqution $\left(\frac{dy}{dx}\right)^{3/2} = \frac{d^2y}{dx^2} + \log\left(\frac{dy}{dx}\right)$

$$\Rightarrow \left(\frac{dy}{dx}\right)^{3/2} - \log\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Squaring both sides, we get $\left(\frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2} + \log\frac{dy}{dx}\right)^2$

The highest order derivative present in equation is $\frac{d^2y}{dx^2}$. Order is 2.

Given differential equation is not polynomial in its derivative, So degree is not defined **[2]**

3. We have ; $\vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}, \vec{b} = -\hat{i} - 2\hat{k}$

$\Rightarrow \quad \vec{a} + \vec{b} = 2\hat{i} - 2\hat{j}, \vec{a} - \vec{b} = 4\hat{i} - 2\hat{j} + 4\hat{k}$ **[½]**

Let θ be the acute angle between the diagonals $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$.

$$\Rightarrow \quad \cos\theta = \frac{(\vec{a} + \vec{b}).(\vec{a} - \vec{b})}{|\vec{a} + \vec{b}|\,|\vec{a} - \vec{b}|}$$

$$= \frac{(2\hat{i} - 2\hat{j}).(4\hat{i} - 2\hat{j} + 4\hat{k})}{\sqrt{8}\,\sqrt{16 + 4 + 16}}$$ **[½]**

$$= \frac{8 + 4}{2\sqrt{2}\,(6)} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$ **[1]**

4. Recall that if the direction ratios of a vector are proportional to a, b, c then its direction cosines are :

$$\pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Therefore, direction cosines of $\vec{r}$ are

$$\pm\frac{2}{\sqrt{2^2 + (-3)^2 + 6^2}}, \pm\frac{-3}{\sqrt{2^2 + (-3)^2 + 6^2}}, \pm\frac{6}{\sqrt{2^2 + (-3)^2 + 6^2}}$$ **[1]**

Since $\vec{r}$ makes an acute angle with x-axis. Therefore, cos α > 0 i.e., $\ell > 0$

So, direction cosines of $\vec{r}$ are $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$

$\therefore \quad \vec{r} = 21\left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$ $\quad \left[\text{Using } \vec{r} = |\vec{r}|(\ell\hat{i} + m\hat{j} + n\hat{k})\right]$

$\Rightarrow \quad \vec{r} = 6\hat{i} - 9\hat{j} + 18\hat{k}$ **[1]**

So, components of $\vec{r}$ along OX, OY and OZ are $6\hat{i}, -9\hat{j}$ and $18\hat{k}$ respectively.

5. For first die ; P(6) = $\frac{1}{2}$, P(6') = $\frac{1}{2}$

$\Rightarrow \quad$ P(1) + P(2) + P(3) + P(4) + P(5) = $\frac{1}{2}$

$\Rightarrow \quad$ P(1) = $\frac{1}{10}$, P(1') = $\frac{9}{10}$

[∵ P(1) = P(2) = P(3) = P(4) = P(5)]

For second die ; P(1) = $\frac{2}{5}$, P(1') = $1 - \frac{2}{5} = \frac{3}{5}$ **[1]**

Let X = Number of one's seen

$\Rightarrow \quad$ P(X = 0) = P(1'). P(1') = $\frac{9}{10} \times \frac{3}{5} = \frac{27}{50}$

P(X = 1) = P(1) . P(1') + P(1') . P(1)

$$= \left(\frac{1}{10}\times\frac{3}{5}\right)+\left(\frac{9}{10}\times\frac{2}{5}\right)=\frac{21}{50}$$

P(X = 2) = P(1).P(1) = $\frac{1}{10}\times\frac{2}{5}=\frac{2}{50}$

Hence ; required probability distribution table is: **[1]**

X	0	1	2
P(X)	$\frac{27}{50}$	$\frac{21}{50}$	$\frac{2}{50}$

6. Since X denotes the number of red balls, then X = 0, 1, 2

P(X = 0) = $\frac{^4C_1}{^7C_1}\times\frac{^3C_1}{^6C_1}=\frac{4}{7}\times\frac{3}{6}=\frac{2}{7}$

P(X = 1) = $\frac{^3C_1}{^7C_1}\times\frac{^4C_1}{^6C_1}+\frac{^4C_1}{^7C_1}\times\frac{^3C_1}{^6C_1}=\frac{3}{7}\times\frac{4}{6}+\frac{4}{7}\times\frac{3}{6}=\frac{2}{7}+\frac{2}{7}=\frac{4}{7}$

P(X = 2) = $\frac{^3C_1}{^7C_1}\times\frac{^2C_1}{^6C_1}=\frac{3}{7}\times\frac{2}{6}=\frac{1}{7}$ **[1]**

Required probability distribution of X is : **[1]**

X	0	1	2
P(X)	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

SECTION – B

7. We have ; $\frac{(x^3-1)}{(x^3+x)}=1-\frac{(x+1)}{(x^3+x)}$

$$= 1-\frac{(x+1)}{x(x^2+1)}$$

Let $\frac{(x+1)}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{(x^2+1)}$ **[½]**

Then ; (x +1) = A(x^2 + 1) + (Bx + C)x

Putting x = 0, we get A = 1

Comparing the coefficients of x, we get C = 1

Comparing the coefficients of x^2, we get

A + B = 0 ⇒ B = –A = –1

∴ A = 1, B = –1 and C = 1

Thus ; $\frac{(x+1)}{x(x^2+1)}=\frac{1}{x}+\frac{(1-x)}{(x^2+1)}$ **[1]**

$\Rightarrow \quad \int \frac{(x^3-1)}{(x^3+x)}dx = \int 1.dx - \int \frac{(x+1)}{x(x^2+1)}dx$ **[½]**

$= x - \left\{ \int \frac{dx}{x} + \int \frac{(1-x)}{(x^2+1)}dx \right\}$

$= x - \int \frac{dx}{x} - \int \frac{dx}{(x^2+1)} + \frac{1}{2}\int \frac{2x}{(x^2+1)}dx$

$= x - \log|x| - \tan^{-1}x + \frac{1}{2}\log|x^2+1| + C$ **[1]**

8. According to the given condition

$$\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x}$$

This is a homogeneous differential equation. Substituting y = vx and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ we get

$v + x\frac{dv}{dx} = v - \cos^2 v \Rightarrow x\frac{dv}{dx} = -\cos^2 v$ **[1]**

$\Rightarrow \quad \int \sec^2 v dv = -\int \frac{dx}{x} \quad \Rightarrow \quad \tan v = -\log x + C$

$\Rightarrow \quad \tan\frac{y}{x} + \log x = C$ **[1]**

Substituting x = 1, $y = \frac{\pi}{4}$, we get. C = 1. Thus, we get

$\tan\frac{y}{x} + \log x = 1$, which is the required equation. **[1]**

OR

Given ; $x\frac{dy}{dx} = y(\log y - \log x + 1)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right)$; **[½]**

which is a homogeneous equation

Put y = vx $\quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \quad v + x\frac{dv}{dx} = v(\log v + 1)$ **[½]**

or $\quad x\frac{dv}{dx} = v\log v$

$\Rightarrow \quad \int \frac{dv}{v\log v} = \int \frac{dx}{x}$ **[½]**

Also, $\frac{1}{v}dv = du$ (Putting log v = u)

$\Rightarrow \quad \int \frac{du}{u} = \int \frac{dx}{x}$ **[½]**

or $\quad \log u = \log x + \log C$

$\Rightarrow \quad u = Cx$

or $\quad \log v = Cx \quad \Rightarrow \log\left(\frac{y}{x}\right) = Cx$ **[1]**

9. Given ; $\vec{a}.\vec{b} = \vec{a}.\vec{c}$ and $\vec{a} \neq \vec{0}$

$\Rightarrow \quad \vec{a}.\vec{b} - \vec{a}.\vec{c} = 0$ and $\vec{a} \neq \vec{0}$

$\Rightarrow \quad \vec{a}.(\vec{b} - \vec{c}) = 0$ and $\vec{a} \neq \vec{0}$

or $\quad \vec{b} - \vec{c} = \vec{0}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

$\Rightarrow \quad \vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$ ……..(1) **[1]**

Again ; $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ and $\vec{a} \neq \vec{0}$

$\Rightarrow \quad (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) = \vec{0}$ and $\vec{a} \neq \vec{0}$

or $\quad \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$ and $\vec{a} \neq \vec{0}$

$\Rightarrow \quad (\vec{b} - \vec{c}) = \vec{0}$ or $\vec{a} \parallel (\vec{b} - \vec{c})$ **[1]**

$\Rightarrow \quad \vec{b} = \vec{c}$ or $\vec{a} \parallel (\vec{b} - \vec{c})$ ……..(2)

From (1) and (2) ; we get $\vec{b} = \vec{c}$ **[1]**

10. Given $\vec{a}.(\vec{b} + \vec{c}) = 0,\ \vec{b}.(\vec{c} + \vec{a}) = 0$ and $\vec{c}.(\vec{a} + \vec{b}) = 0$

$|\vec{a} + \vec{b} + \vec{c}|^2 = (\vec{a} + \vec{b} + \vec{c}).(\vec{a} + \vec{b} + \vec{c})$ **[$\because \vec{a}.\vec{a} = |\vec{a}|^2$]** **[1]**

$= \vec{a}.\vec{a} + \vec{a}.(\vec{b} + \vec{c}) + \vec{b}.\vec{b} + \vec{b}.(\vec{a} + \vec{c}) + \vec{c}.\vec{c} + \vec{c}.(\vec{a} + \vec{b})$ **[1]**

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 = 9 + 16 + 25 = 50$

Therefore, $|\vec{a} + \vec{b} + \vec{c}| = \sqrt{50} = 5\sqrt{2}$ **[1]**

OR

Comparing the given equations with the standard equations :

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$; we get

$\vec{a}_1 = (\hat{i} - \hat{j}), \quad \vec{b}_1 = (2\hat{i} + \hat{k})$

$\vec{a}_2 = (2\hat{i} - \hat{j})$ and $\vec{b}_2 = (\hat{i} + \hat{j} - \hat{k})$ **[1]**

$\Rightarrow \quad (\vec{a}_2 - \vec{a}_1) = \hat{i}$ and $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$= (0-1)\hat{i} - (-2-1)\hat{j} + (2-0)\hat{k}$

$= -\hat{i} + 3\hat{j} + 2\hat{k}$

$\Rightarrow \quad \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{14}$ **[½]**

Shortest distance = $\left|\dfrac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{\left|\vec{b}_1 \times \vec{b}_2\right|}\right|$

$= \left|\dfrac{\hat{i}.(-\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{14}}\right| = \dfrac{1}{\sqrt{14}} \neq 0$ **[1]**

Since the shortest distance between the given lines is not zero ; the given lines do not intersect. **[½]**

SECTION – C

11. Let $I = \int_0^{\pi/2} \dfrac{\sin^2 x}{(1 + \sin x . \cos x)} dx$ ....(1)

$\Rightarrow \quad I = \int_0^{\pi/2} \dfrac{\sin^2\left(\dfrac{\pi}{2} - x\right)}{1 + \sin\left(\dfrac{\pi}{2} - x\right).\cos\left(\dfrac{\pi}{2} - x\right)} dx$ **[½]**

$\therefore \quad I = \int_0^{\pi/2} \dfrac{\cos^2 x}{1 + \cos x . \sin x} dx$ ....(2)

Adding (1) and (2) ; we get

$2I = \int_0^{\pi/2} \dfrac{(\sin^2 x + \cos^2 x)}{1 + \sin x . \cos x} dx$ **[½]**

$= \int_0^{\pi/2} \dfrac{dx}{1 + \sin x . \cos x}$

$\Rightarrow 2I = \int_0^{\pi/2} \dfrac{\sec^2 x \, dx}{\sec^2 x + \tan x}$

$= \int_0^{\pi/2} \dfrac{\sec^2 x}{(1 + \tan^2 x) + \tan x} dx$ **[1]**

Let tanx = t $\Rightarrow$ $\sec^2 x \, dx = dt$

When x = 0, t = 0 ; x = $\dfrac{\pi}{2}$, t = ∞

$\Rightarrow \quad 2I = \int_0^{\infty} \dfrac{dt}{(t^2 + t + 1)}$

$= \int_0^{\infty} \dfrac{dt}{\left(t + \dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2}$

$$= \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^{\infty}$$ **[1]**

$$\Rightarrow \quad 2I = \frac{2}{\sqrt{3}}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3}\right) = \frac{\pi}{3\sqrt{3}}$$ **[1]**

12. The given curve is $y = 2x - x^2$

$\Rightarrow \quad (x^2 - 2x + 1) = (-y + 1)$

or $\quad (x - 1)^2 = -1(y - 1)$

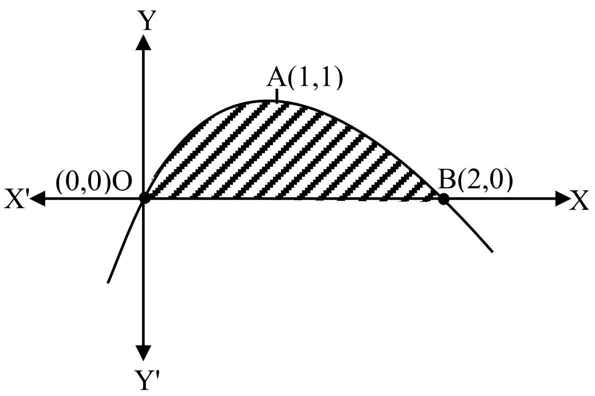
$\Rightarrow \quad X^2 = -Y$

Also, $y = 0$ (x-axis)

$\Rightarrow \quad 2x - x^2 = 0 \quad$ or $\quad x(2 - x) = 0$

$\Rightarrow \quad x = 0 \quad$ or $\quad x = 2$ **[1]**

Thus ; the curve intersect x-axis at O(0, 0) and B(2, 0)



[2]

$\therefore$ Required area $= \int_0^2 y\,dx$

$$= \int_0^2 (2x - x^2)\,dx$$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$

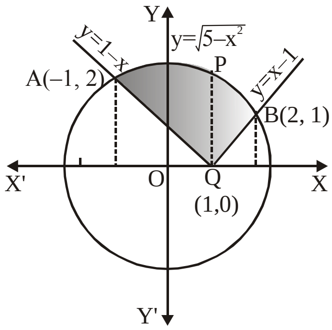
$$= \left(4 - \frac{8}{3}\right)\text{sq. units}$$

$$= \frac{4}{3}\text{sq. units}$$ **[1]**

OR

The required area is above the curve $y = |x-1| = \begin{cases} x-1, \text{ for } x \ge 1 \\ -(x-1), \text{ for } x < 1 \end{cases}$, which is a pair of half rays and below the curve $y = \sqrt{5-x^2}$, which is the upper half (above X-axis) of the circle $x^2 + y^2 = 5$, whose centre is (0, 0) and radius $\sqrt{5}$.

The two curves meet where



$|x-1| = \sqrt{5-x^2}$

$\Rightarrow \quad (x-1)^2 = 5 - x^2$ **[1]**

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad x = -1, x = 2.$

When $x = -1$, then $y = 2$

and when $x = 2$, then $y = 1$.

So, the two curves meet in the points A(–1, 2) and B(2, 1). **[1]**

Required area is shown shaded in the figure and is equal to = area AQPA + area PQBP

$$= \int_{-1}^{1}\left[\sqrt{5-x^2} - (1-x)\right]dx + \int_{1}^{2}\left[\sqrt{5-x^2} - (x-1)\right]dx$$

$$= \left(\int_{-1}^{1}\sqrt{5-x^2}\,dx + \int_{1}^{2}\sqrt{5-x^2}\,dx\right) - \int_{-1}^{1}(1-x)dx - \int_{1}^{2}(x-1)dx$$

$$= \int_{-1}^{2}\sqrt{5-x^2}\,dx - \int_{-1}^{1}(1-x)dx - \int_{1}^{2}(x-1)dx$$

$$= \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right]_{-1}^{2} - \left[x - \frac{x^2}{2}\right]_{-1}^{1} - \left[\frac{x^2}{2} - x\right]_{1}^{2}$$ **[1]**

$$= \left\{\frac{2}{2}\sqrt{5-4} + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right\} - \left\{\frac{-1}{2}\sqrt{5-1} + \frac{5}{2}\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right\} - \left(\frac{1}{2}+\frac{3}{2}\right) - \left(0 + \frac{1}{2}\right)$$

$$= 1 + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) + 1 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \frac{5}{2}$$

$$= -\frac{1}{2} + \frac{5}{2}\left\{\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) + \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right\} \qquad \left(\because \sin^{-1} x = \cos^{-1}\sqrt{1-x^2} \text{ for } 0 \le x \le 1\right)$$

$$= -\frac{1}{2} + \left(\frac{5}{2} \times \frac{\pi}{2}\right) = \left(-\frac{1}{2} + \frac{5\pi}{4}\right) \text{ square units.}$$ **[1]**

13. Let γ be the angle made by $\vec{n}$ with z-axis. Then direction cosines of $\vec{n}$ are

$l = \cos 45° = \frac{1}{\sqrt{2}}$, $m = \cos 60° = \frac{1}{2}$ and $n = \cos\gamma$

$\therefore l^2 + m^2 + n^2 = 1$ **[1]**

$\Rightarrow \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + n^2 = 1$

$\Rightarrow\ n^2 = \frac{1}{4} \Rightarrow n = \frac{1}{2}$ $\quad [\because \gamma \text{ is acute } \therefore n = \cos\gamma > 0]$ **[1]**

We have, $|\vec{n}| = 8$

$\therefore\ \vec{n} = |\vec{n}|(\ell\hat{i} + m\hat{j} + n\hat{k})$

$\Rightarrow\ \vec{n} = 8\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}\right) = 4\sqrt{2}\hat{i} + 4\hat{j} + 4\hat{k}$ **[1]**

The required plane passes through the point $(\sqrt{2}, -1, 1)$ having position vector $\vec{a} = \sqrt{2}\,\hat{i} - \hat{j} + \hat{k}$.

Hence its equation in vector form is given as $\left[\vec{r} - (\sqrt{2}\,\hat{i} - \hat{j} + \hat{k})\right].(4\sqrt{2}\,\hat{i} + 4\hat{j} + 4\hat{k}) = 0$ **[1]**

CASE BASED / DATA BASED

14. (i) Let D_1 denote the event that the patient has disease d_1. The events D_2 and D_3 are defined similarly.

Then, $P(D_1) = \frac{3200}{10000} = 0.32,$

$P(D_2) = \frac{3500}{10000} = 0.35,$

and $P(D_3) = \frac{3300}{10000} = 0.33,$

Further, $P(S \mid D_1) = \frac{P(S \cap D_1)}{P(D_1)} = \frac{3100}{3200} = 0.97 \text{ (approx.)}$

$P(S \mid D_2) = \frac{3300}{3500} = 0.94 \text{ (approx.)}$

and $P(S \mid D_3) = \frac{3000}{3300} = 0.91 \text{ (approx.)}$ **[1]**

Now $P(S) = P(D_1)\,P(S|D_1) + P(D_2)\,P(S|D_2) + P(D_3)\,P(S|D_3)$

$= (0.32 \times 0.97) + (0.35 \times 0.94) + (0.33 \times 0.91)$

$= 0.3104 + 0.329 + 0.3003$

$= 0.9397$ **[1]**

(ii) Using Baye's theorem, we get

$P(D_1|S)$ = the probability that the patient has disease d_1 knowing that he/she has symptoms S.

$$= \frac{P(D_1)P(S \mid D_1)}{P(D_1)P(S \mid D_1) + P(D_2)P(S \mid D_2) + P(D_3)P(S \mid D_3)}$$

$$= \frac{0.32 \times 0.97}{(0.32 \times 0.97) + (0.35 \times 0.94) + (0.33 \times 0.91)}$$ **[1]**

$$= \frac{0.3104}{0.3104 + 0.329 + 0.3003}$$

$= \frac{0.3104}{0.9397} = 0.33$ approx. **[1]**