SAMPLE QUESTION PAPER (SOLUTION)

CLASS: XII

Session: 2021-22

Mathematics (Code-041)

Term - 2

SECTION – A

1. $\int \frac{x^3+x}{x^4-9}dx = \int \frac{x^3}{x^4-9}dx + \int \frac{x}{x^4-9}dx = \frac{1}{4}\int \frac{4x^3}{x^4-9}dx + \frac{1}{2}\int \frac{2x}{x^4-9}dx$ **[½]**

$= \frac{1}{4}\log|x^4-9| + \frac{1}{2}\int \frac{dt}{t^2-9}$ [Where $t = x^2$]

$= \frac{1}{4}\log|x^4-9| + \frac{1}{2}\cdot\frac{1}{6}\log\left|\frac{t-3}{t+3}\right| + C$ **[½]**

$= \frac{1}{4}\log|x^4-9| + \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C$ **[1]**

OR

$\int \frac{e^x(x+1)}{(x+3)^3}dx = \int e^x\left\{\frac{(x+3)-2}{(x+3)^3}\right\}dx$

$= \int e^x\left\{\frac{1}{(x+3)^2} - \frac{2}{(x+3)^3}\right\}dx$ **[½]**

$= \int e^x\{f(x)+f'(x)\}\,dx \left[\text{where } f(x) = \frac{1}{(x+3)^2} \Rightarrow f'(x) = -\frac{2}{(x+3)^3}\right]$ **[½]**

$= e^x f(x) + C = \frac{e^x}{(x+3)^2} + C$ **[1]**

2. Since it is not a polynomial w.r.t. the derivatives hence its degree is not defined but its order is 4. **[2]**

3. Given $\vec{a}$ & $\vec{b}$ are unit vectors i.e. $|\vec{a}| = 1 = |\vec{b}|$

and $|\sqrt{3}\vec{a}-\vec{b}| = 1 \Rightarrow |\sqrt{3}\vec{a}-\vec{b}|^2 = 1$ **[½]**

$\Rightarrow (\sqrt{3}\vec{a}-\vec{b}).(\sqrt{3}\vec{a}-\vec{b}) = 1$

or $3|\vec{a}|^2 - \sqrt{3}\vec{a}.\vec{b} - \sqrt{3}\vec{b}.\vec{a} + |\vec{b}|^2 = 1$ $(\because \vec{a}.\vec{a} = |\vec{a}|^2)$

$\Rightarrow 3 - 2\sqrt{3}\vec{a}.\vec{b} + 1 = 1$ $(\because \vec{a}.\vec{b} = \vec{b}.\vec{a})$

or $\vec{a}.\vec{b} = \frac{\sqrt{3}}{2} \Rightarrow |\vec{a}||\vec{b}|\cos\theta = \frac{\sqrt{3}}{2}$ $(\because \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta)$ **[½]**

$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$ **[1]**

4. Given line is $\frac{x-1}{3}=\frac{y+4}{7}=\frac{z+4}{2}=\lambda$ (let)

The coordinates of arbitary point lying on line is $(3\lambda + 1, 7\lambda - 4, 2\lambda - 4)$ **[½]**

$\because$ Line cuts the xy-plane so its z-coordinate will be zero.

$\therefore$ $2\lambda - 4 = 0 \Rightarrow \lambda = 2$ **[½]**

Req. point is (7, 10, 0) **[1]**

5. Let A ≡ event of selecting first purse

B ≡ event of selecting second purse

C ≡ event of drawing a copper coin

Then given event has two disjoint cases: AC and BC

$\therefore$ $P(C) = P(AC+BC) = P(AC)+P(BC) = P(A)P(C/A)+P(B)P(C/B)$ **[1]**

$$=\left(\frac{1}{2}.\frac{4}{7}\right)+\left(\frac{1}{2}.\frac{6}{8}\right)=\frac{37}{56}$$ **[1]**

6. Let E : Obtaining the sum 8 on the dice

F : Red die resulted in a number less than 4.

i.e. E = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}

$\Rightarrow$ n(E) = 5

F = {(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(6,1),(6,2),(6,3)}

$\Rightarrow$ n(F) = 18

and $E \cap F$ = {(5, 3), (6, 2)}

$\Rightarrow n(E \cap F) = 2$

Here, $P(F)=\frac{18}{36}=\frac{1}{2}$ **[½]**

and $P(E\cap F)=\frac{2}{36}=\frac{1}{18}$ **[½]**

$\therefore$ Req. probability = P(E/F)

$$=\frac{P(E\cap F)}{P(F)}=\frac{1/18}{1/2}=\frac{1}{9}$$ **[1]**

SECTION – B

7. $I=\int\frac{x^2+x+1}{(x^2+1)(x+2)}dx$

Let $\frac{x^2+x+1}{(x+2)(x^2+1)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+1}$ .... (1) **[1]**

$x^2 + x + 1 = A(x^2 + 1) + (Bx + C)(x + 2)$

At $x = -2$, $A=\frac{3}{5}$

and at $x = 0$, $A + 2C = 1 \Rightarrow C=\frac{1}{5}$

Equating the coeff. of x^2,

$1 = A + B \Rightarrow B=1-\frac{3}{5}=\frac{2}{5}$ **[½]**

Put in equation (1) and integrate

$$\therefore I=\int\frac{x^2+x+1}{(x+2)(x^2+1)}dx=\frac{3}{5}\int\frac{1}{x+2}dx+\int\frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1}dx$$ **[½]**

$$=\frac{3}{5}\int\frac{1}{x+2}dx+\frac{1}{5}\int\frac{2x}{x^2+1}dx+\frac{1}{5}\int\frac{1}{x^2+1}dx$$

$$=\frac{3}{5}\log(x+2)+\frac{1}{5}\log(x^2+1)+\frac{1}{5}\tan^{-1}x+C$$ **[1]**

8. The given differential equation can be written as $\frac{dy}{dx}=\frac{y\cos\left(\frac{y}{x}\right)+x}{x\cos\left(\frac{y}{x}\right)}$ ...(i) [½]

It is a differential equation of the form $\frac{dy}{dx}$ = F(x, y).

Here $F(x,\ y)=\frac{y\cos\left(\frac{y}{x}\right)+x}{x\cos\left(\frac{y}{x}\right)}$

Replacing x by λx and y by λy, we get $F(\lambda x,\ \lambda y)=\frac{\lambda\left[y\cos\left(\frac{y}{x}\right)+x\right]}{\lambda\left(x\cos\frac{y}{x}\right)}=\lambda^0[F(x,\ y)]$ **[½]**

Thus, F(x, y) is a homogeneous function of degree zero.
Therefore, the given differential equation is a homogeneous differential equation.
To solve it we make the substitution

$y = vx$ ...(ii)

Differentiating equation (ii) with respect to x, we get

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \quad ...(iii)$$ **[½]**

Substituting the value of y and $\frac{dy}{dx}$ in equation (i), we get

$$v+x\frac{dv}{dx}=\frac{v\cos v+1}{\cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{v\cos v+1}{\cos v}-v$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1}{\cos v}$$

$$\Rightarrow \quad \cos v\,dv=\frac{1}{x}dx$$ **[½]**

Therefore $\int\cos v\ dv=\int\frac{1}{x}dx$

$\Rightarrow \quad \sin v = \log|x| + \log|C|$

$\Rightarrow \quad \sin v = \log|Cx|$

Replacing v by $\frac{y}{x}$, we get $\sin\left(\frac{y}{x}\right)=\log|Cx|$ **[1]**

which is the general solution of the differential equation (i).

OR

$$\frac{dy}{dx} = -\frac{x + y\cos x}{1+\sin x}$$

$$\frac{dy}{dx} + \left(\frac{\cos x}{1+\sin x}\right)y = -\frac{x}{1+\sin x} \quad .....(i)$$ [½]

Eq. (i) is a linear diff. eq. of the form $\frac{dy}{dx} + Py = Q$,

where $P = \frac{\cos x}{1+\sin x}$ and $Q = -\frac{x}{1+\sin x}$

$\Rightarrow$ I.F. $= e^{\int P dx} = e^{\int \frac{\cos x}{1+\sin x}dx}$

$= e^{\log(1+\sin x)} = 1 + \sin x.$ [1]

Hence, solution of diff. eq. (i) is

$y \times \text{I.F.} = \int (Q \times \text{I.F.})dx$

$\Rightarrow \quad y(1 + \sin x) = \int -\left(\frac{x}{1+\sin x}\right) \times (1+\sin x)dx$ [½]

$\Rightarrow \quad y(1 + \sin x) = -\frac{x^2}{2} + C$ [½]

When x = 0, y = 1 $\Rightarrow$ C = 1

$\therefore \quad y(1+\sin x) = -\frac{x^2}{2} + 1$ [½]

9. Let vector $\vec{c} = a\hat{i} + b\hat{j} + c\hat{k}$

Given $\vec{a} \times \vec{c} = \vec{b}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} = 0\hat{i} + \hat{j} - \hat{k}$$

$\Rightarrow \quad \hat{i}(c-b) - \hat{j}(c-a) + \hat{k}(b-a) = 0\hat{i} + \hat{j} - \hat{k}$ [1]

Equating the coefficient of both the sides ;

c – b = 0 ....(i), a – c = 1 ....(ii), b – a = –1 ....(iii) [½]

Now $\vec{a} \cdot \vec{c} = 3$

$(\hat{i} + \hat{j} + \hat{k}).(a\hat{i} + b\hat{j} + c\hat{k}) = 3$

a + b + c = 3 ....(iv) [½]

On solving equation (i), (ii), (iii), (iv), we get

$a = \frac{5}{3}, b = c = \frac{2}{3}$

So the required vector $\vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$ [1]

10. $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$

$\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + 2\mu(2\hat{i} + 3\hat{j} + 4\hat{k})$

Lines are parallel

Here $\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{a}_2 = 2\hat{i} + 4\hat{j} + 5\hat{k}$

$\vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore \quad \vec{a}_2 - \vec{a}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$ and $|\vec{b}| = \sqrt{4+9+16} = \sqrt{29}$ **[1]**

$\Rightarrow \quad \vec{b} \times (\vec{a}_2 - \vec{a}_1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 2 \end{vmatrix}$

$= \hat{i}(6-8) - \hat{j}(4-4) + \hat{k}(4-3) = -2\hat{i} + \hat{k}$ **[1]**

$\therefore \quad |\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = \sqrt{4+1} = \sqrt{5}$

$\Rightarrow$ Req. Distance $= \dfrac{|\vec{b} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}|} = \sqrt{\dfrac{5}{29}}$ units **[1]**

OR

Let $P_1 \Rightarrow 2x + y - z - 3 = 0$

$P_2 \Rightarrow 5x - 3y + 4z + 9 = 0$

Required plane is $P_1 + \lambda P_2 = 0$

$\Rightarrow \quad (2x + y - z - 3) + \lambda(5x - 3y + 4z + 9) = 0$

$\Rightarrow \quad (2 + 5\lambda)x + (1 - 3\lambda)y + (-1 + 4\lambda)z - 3 + 9\lambda = 0$ ....(1) **[1]**

Plane (1) is parallel to the line $\dfrac{x-1}{2} = \dfrac{y-3}{4} = \dfrac{z-5}{5}$

i.e. Normal of plane (1) is perpendicular to the line

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow \quad (2 + 5\lambda) \times 2 + (1 - 3\lambda) \times 4 + (-1 + 4\lambda) \times 5 = 0$

$\Rightarrow \quad \lambda = -\dfrac{1}{6}$ **[1]**

So, equation of req. plane is $\left(2 - \dfrac{5}{6}\right)x + \left(1 + \dfrac{1}{2}\right)y + \left(-1 - \dfrac{2}{3}\right)z - 3 - \dfrac{3}{2} = 0$

$\Rightarrow \quad \dfrac{7}{6}x + \dfrac{3}{2}y - \dfrac{5}{3}z - \dfrac{9}{2} = 0$

$\Rightarrow \quad 7x + 9y - 10z - 27 = 0$ **[1]**

SECTION – C

11. Let $I = \int_{-2}^{2} \frac{x^2}{1+5^x} dx$ ....(i)

$\Rightarrow \quad I = \int_{-2}^{2} \frac{x^2}{1+5^{-x}} dx \qquad \left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$ **[1]**

or $\quad I = \int_{-2}^{2} \frac{x^2 5^x}{1+5^x} dx$ ....(ii) **[1]**

Equation (i) + (ii), gives

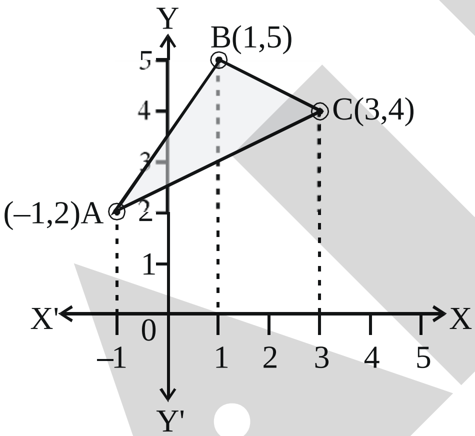
$$2I = \int_{-2}^{2} x^2 dx$$

$\Rightarrow \quad 2I = 2\int_0^2 x^2 dx \qquad \left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is an even fn.}\right]$ **[1]**

$\Rightarrow \quad I = \int_0^2 x^2 dx = \frac{1}{3}(x^3)_0^2 = \frac{1}{3}(2^3 - 0^3)$

$\therefore \quad I = \frac{8}{3}$ **[1]**

12. **[1]**



Eq. of line AB is $y = \frac{3}{2}x + \frac{7}{2}$, Eq. of line BC is $y = \frac{-x}{2} + \frac{11}{2}$, Eq. of line AC is $y = \frac{x}{2} + \frac{5}{2}$ **[1]**

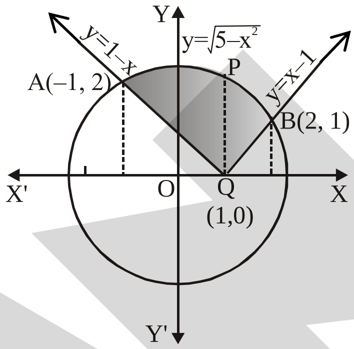
Req. Area $= \int_{-1}^{1}\left(\frac{3}{2}x + \frac{7}{2}\right)dx + \int_{1}^{3}\left(-\frac{x}{2} + \frac{11}{2}\right)dx - \int_{-1}^{3}\left(\frac{x}{2} + \frac{5}{2}\right)dx$ **[½]**

$= \left(\frac{3x^2}{4} + \frac{7x}{2}\right)_{-1}^{1} + \left(-\frac{x^2}{4} + \frac{11x}{2}\right)_{1}^{3} - \left(\frac{x^2}{4} + \frac{5x}{2}\right)_{-1}^{3}$ **[½]**

= (7 + 9 – 12) = 4 square units **[1]**

OR

The required area is above the curve $y = |x-1| = \begin{cases} x-1, \text{ for } x \ge 1 \\ -(x-1), \text{ for } x < 1 \end{cases}$, which is a pair of half rays above and below the curve $y = \sqrt{5-x^2}$, which is the upper half (above X-axis) of the circle $x^2 + y^2 = 5$, whose centre is (0, 0) and radius $\sqrt{5}$.



The two curves meet where $|x-1| = \sqrt{5-x^2}$

$\Rightarrow \quad (x-1)^2 = 5 - x^2$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad x = -1, x = 2.$ **[1½]**

When $x = -1$, then $y = 2$

and when $x = 2$, then $y = 1$.

So, the two curves meet in the points A(–1, 2) and B(2, 1). **[½]**

Required area is shown shaded in the figure and is equal to = area AQPA + area PQBP

$$= \int_{-1}^{1}\left[\sqrt{5-x^2} - (1-x)\right]dx + \int_{1}^{2}\left[\sqrt{5-x^2} - (x-1)\right]dx$$ **[½]**

$$= \left(\int_{-1}^{1}\sqrt{5-x^2}\,dx + \int_{1}^{2}\sqrt{5-x^2}\,dx\right) - \int_{-1}^{1}(1-x)dx - \int_{1}^{2}(x-1)\,dx$$

$$= \int_{-1}^{2}\sqrt{\left(\sqrt{5}\right)^2 - x^2}\,dx - \int_{-1}^{1}(1-x)dx - \int_{1}^{2}(x-1)dx$$

$$= \left[\frac{x}{2}\sqrt{\left(\sqrt{5}\right)^2 - x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right]_{-1}^{2} - \left[x - \frac{x^2}{2}\right]_{-1}^{1} - \left[\frac{x^2}{2} - x\right]_{1}^{2}$$ **[½]**

$$= \left\{\frac{2}{2}\sqrt{5-4} + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right\} - \left\{\frac{-1}{2}\sqrt{5-1} + \frac{5}{2}\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right\} - \left(\frac{1}{2} + \frac{3}{2}\right) - \left(0 + \frac{1}{2}\right)$$

$$= 1 + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) + 1 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \frac{5}{2}$$

$$= -\frac{1}{2} + \frac{5}{2}\left\{\sin^{-1}\frac{2}{\sqrt{5}} + \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right\} \qquad (\because \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} \text{ for } 0 \le x \le 1)$$

$$= -\frac{1}{2} + \frac{5}{2} \times \frac{\pi}{2} = -\frac{1}{2} + \frac{5\pi}{4} \text{ square units.}$$ **[1]**

13. Required plane is given as $P_1 + \lambda P_2 = 0$

$\Rightarrow \quad \left[\vec{r}.(\hat{i} - 2\hat{j} + 3\hat{k}) - 4\right] + \lambda\left[\vec{r}.(-2\hat{i} + \hat{j} + \hat{k}) + 5\right] = 0$

$\Rightarrow \quad \vec{r}.\left[(1-2\lambda)\hat{i} + (-2+\lambda)\hat{j} + (3+\lambda)\hat{k}\right] = 4 - 5\lambda \qquad$ ....(1) **[1]**

Now : x-axis intercept = y-axis intercept

$\Rightarrow \quad \frac{4-5\lambda}{1-2\lambda} = \frac{4-5\lambda}{-2+\lambda}$ **[1]**

$\therefore \quad 1 - 2\lambda = -2 + \lambda$

$\Rightarrow \quad 3\lambda = 3$

or $\quad \lambda = 1$

[Putting this value of λ in equation (1) ; we get] **[1]**

$\vec{r}.(-\hat{i}-\hat{j}+4\hat{k}) = -1$

$\vec{r}.(-\hat{i}-\hat{j}+4\hat{k})+1 = 0$ **[1]**

SECTION – D

14. Given that

E_1 = student knows the answer

E_2 = student guesses the answer

E = Answer correctly

$P(E_1) = \frac{3}{5}$ and $P(E_2) = \frac{2}{5}$

$P(E/E_1) = 1$ and $P(E/E_2) = \frac{1}{3}$

(i) $\sum_{k=1}^{k=2} P(E|E_k)\, P(E_k)$

$= P(E/E_1) \,.\, P(E_1) + P(E/E_2) \,.\, P(E_2)$

$= 1 \times \frac{3}{5} + \frac{1}{3} \times \frac{2}{5} = \frac{9+2}{15} = \frac{11}{15}$ Ans. **[2]**

(ii) $P(E_1/E) = \frac{P(E_1).P(E/E_1)}{P(E_1)\,P(E/E_1) + P(E_2).P(E/E_2)}$

$= \frac{\frac{3}{5}\times 1}{\frac{3}{5}\times 1 + \frac{2}{5}\times\frac{1}{3}}$

$= \frac{3/5}{11/15}$

$= \frac{9}{11}$ **[2]**