खण्ड–A / SECTION–A

1. $\int \sqrt{1+\sin\left(\frac{x}{4}\right)}\,dx$

$=\int \sqrt{\left(\sin^2\frac{x}{8}+\cos^2\frac{x}{8}\right)+\left(2\sin\frac{x}{8}\cos\frac{x}{8}\right)}\,dx$ [½]

$=\int \sqrt{\left(\sin\frac{x}{8}+\cos\frac{x}{8}\right)^2}\,dx=\int\left(\sin\frac{x}{8}+\cos\frac{x}{8}\right)dx$ [½]

$=\frac{-\cos\frac{x}{8}}{\left(\frac{1}{8}\right)}+\frac{\sin\frac{x}{8}}{\left(\frac{1}{8}\right)}+c=8\left(\sin\frac{x}{8}-\cos\frac{x}{8}\right)+C$ [1]

OR

$\int\frac{dx}{\cos^3x\sqrt{2\sin 2x}}=\int\frac{dx}{\cos^3x\sqrt{4\sin x\cos x}}$

$=\frac{1}{2}\int\frac{dx}{\cos^{7/2}x\,\sin^{1/2}x}$

$=\frac{1}{2}\int\frac{\sec^4 x}{\sqrt{\tan x}}dx=\frac{1}{2}\int\frac{(1+\tan^2x)\sec^2x}{\sqrt{\tan x}}dx$ [½]

$=\frac{1}{2}\int\frac{1+t^2}{\sqrt{t}}dt$ (Put $\tan x=t,\ \therefore \sec^2 x\,dx=dt$)

$=\frac{1}{2}\int t^{-1/2}dt+\frac{1}{2}\int t^{3/2}dt=t^{1/2}+\frac{t^{5/2}}{5}+C$ [½]

$=\sqrt{\tan x}+\frac{1}{5}\tan^{5/2}x+C$ [1]

2. Making fourth power both the sides, we get the differential equation

$\left(\frac{d^2y}{dx^2}\right)^4=y+\left(\frac{dy}{dx}\right)^2$ [½]

Obviously, order is 2 and degree is 4. [½]

Hence, sum of order and degree is 2 + 4 = 6 [1]

3. $|\hat{x}-\hat{y}|^2=(\hat{x}-\hat{y}).(\hat{x}-\hat{y})=1+1-2|\hat{x}||\hat{y}|\cos\pi$ $[\because |\hat{x}|^2=|\hat{y}|^2=1, |\hat{x}|=|\hat{y}|=1]$

$=2-2\cos\pi,\ \therefore |\hat{x}-\hat{y}|^2=4$ [1]

So, $\frac{1}{2}|\hat{x}-\hat{y}|=1$ [1]

4. Here, $\ell = \cos\theta, m = \cos\beta, n = \cos\theta$, $(\because \ell = n)$

Now, $\ell^2 + m^2 + n^2 = 1 \Rightarrow 2\cos^2\theta + \cos^2\beta = 1$ **[½]**

$\Rightarrow 2\cos^2\theta = \sin^2\beta$

Given, $\sin^2\beta = 3\sin^2\theta \Rightarrow 2\cos^2\theta = 3\sin^2\theta$ **[½]**

$\Rightarrow 5\cos^2\theta = 3, \therefore \cos^2\theta = \frac{3}{5}.$ **[1]**

5. Given : E = {X is a prime number}

$P(E) = P(2) + P(3) + P(5) + P(7) = 0.62,$ **[½]**

F = { X < 4}, $P(F) = P(1) + P(2) + P(3) = 0.50$

and $P(E \cap F) = P(2) + P(3) = 0.35$ **[½]**

$\therefore P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$= 0.62 + 0.50 - 0.35 = 0.77.$ **[1]**

6. Let A = The event that atleast one girl will be chosen,

B = The event that exactly 2 girls will be chosen.

Now, P(A) = 1 – $P(\bar{A})$ = 1 – P (no girl is chosen)

= 1 – P(4 boys are chosen)

$= 1 - \frac{{}^8C_4}{{}^{12}C_4} = 1 - \frac{14}{99} = \frac{85}{99}$ **[½]**

and $P(A \cap B) = P(\text{2 boys and 2 girls are chosen})$

$= \frac{{}^8C_2 \times {}^4C_2}{{}^{12}C_4} = \frac{56}{165}$ **[½]**

Hence, $P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{56/165}{85/99} = \frac{168}{425}$ **[1]**

खण्ड–B / SECTION–B

7. Let $x^2 = y$, then

$$\frac{x^2+1}{(x^2+2)(2x^2+1)} = \frac{y+1}{(y+2)(2y+1)}$$

Now, let $\frac{y+1}{(y+2)(2y+1)} = \frac{A}{y+2} + \frac{B}{2y+1}$ ...(1) **[½]**

$\Rightarrow y + 1 = A(2y + 1) + B(y + 2)$

Put $y = -2 \Rightarrow A = 1/3$

Put $y = -\frac{1}{2} \Rightarrow B = \frac{1}{3}$

$\therefore$ From equation (1), $\frac{y+1}{(y+2)(2y+1)} = \frac{1}{3} \cdot \frac{1}{y+2} + \frac{1}{3} \cdot \frac{1}{2y+1}$

Replacing y by x^2, we get $\frac{x^2+1}{(x^2+2)(2x^2+1)} = \frac{1}{3}\cdot\left(\frac{1}{x^2+2}\right) + \frac{1}{3(2x^2+1)}$ **[1]**

$\therefore \quad I = \int \frac{x^2+1}{(x^2+2)(2x^2+1)}dx$

$= \frac{1}{3}\int \frac{1}{x^2+2}dx + \frac{1}{3}\int \frac{1}{(\sqrt{2}x)^2+1}dx$ **[½]**

$\Rightarrow \quad I = \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{3\sqrt{2}}\tan^{-1}\left(\sqrt{2}x\right) + C$ **[1]**

8. Given $\frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right)$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ **[½]**

$v + x\frac{dv}{dx} = v - \sin^2 v \Rightarrow -\text{cosec}^2 v\, dv = \frac{dx}{x}$ **[½]**

Integrating both sides, $-\int \text{cosec}^2 v\, dv = \int \frac{dx}{x}$ **[½]**

$\Rightarrow \cot v = \log x + c$

$\Rightarrow \cot\frac{y}{x} = \log x + c$

This curve passes through the point $\left(1, \frac{\pi}{4}\right)$

$\therefore c = 1 \Rightarrow \cot\frac{y}{x} = \log_e x + \log_e e$ **[½]**

$\cot\frac{y}{x} = \log x e \Rightarrow y = x\cot^{-1}(\log x e)$. **[1]**

OR

Rearranging the terms, $\frac{dy}{dt} - \frac{t}{1+t}y = \frac{1}{1+t}$ **[½]**

I.F. $= e^{\int -\frac{t}{1+t}dt} = e^{-t}.(1+t)$ **[½]**

$\therefore$ Solution is y(I.F.) = $\int Q.(\text{I.F.})dt + c$

$\Rightarrow ye^{-t}.(1+t) = \int (1+t).e^{-t}\frac{1}{(1+t)} + c$ **[1]**

$\Rightarrow \quad ye^{-t}(1+t) = -e^{-t} + c$

Also, $y(0) = -1 \Rightarrow c = 0 \Rightarrow y(1) = \frac{-1}{2}$. **[1]**

9. If angle between $\vec{b}$ and $\vec{c}$ is α and $|\vec{b}\times\vec{c}|=\sqrt{15}$

$$\Rightarrow \quad |\vec{b}|\;|\vec{c}|\;\sin\alpha = \sqrt{15}$$

$$\sin\alpha=\frac{\sqrt{15}}{4}\;;\;\therefore\;\cos\alpha=\frac{1}{4} \qquad \left[\because |\vec{b}|=4, |\vec{c}|=1\right]$$ **[1]**

$$\vec{b}-2\vec{c} = \lambda\vec{a} \Rightarrow |\vec{b}-2\vec{c}|^2=\lambda^2|\vec{a}|^2$$

$$|\vec{b}|^2+4|\vec{c}|^2-4\left(\vec{b}.\vec{c}\right)=\lambda^2|\vec{a}|^2$$ **[1]**

$$16+4-4\{|\vec{b}||\vec{c}|\cos\alpha\}=\lambda^2 \qquad \left[\because |\vec{a}|=1\right]$$

$$16+4-4\times4\times1\times\frac{1}{4}=\lambda^2\Rightarrow\lambda^2=16\Rightarrow\lambda=\pm4.$$ **[1]**

10. Equation of plane passing through the point (1,0,–1) is,

$$a(x-1)+b(y-0)+c(z+1)=0 \qquad .....(i)$$ **[½]**

Also, plane (i) is passing through (3, 2, 2)

$\therefore \quad a(3-1)+b(2-0)+c(2+1)=0$

or $\quad 2a+2b+3c=0 \qquad .....(ii)$ **[½]**

Plane (i) is also parallel to the line $\frac{x-1}{2}=\frac{y-1}{-2}=\frac{z-2}{3}$

$\therefore 2a-2b+3c=0 \qquad .....(iii)$

From (ii) and (iii), $\frac{a}{-3}=\frac{b}{0}=\frac{c}{2}$ **[1]**

Therefore, the required plane is,

$$-3(x-1)+0(y-0)+2(z+1)=0$$

or $\quad -3x+2z+5=0$ **[1]**

OR

$$\text{S.D.}=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\left(b_1c_2-b_2c_1\right)^2+\left(c_1a_2-c_2a_1\right)^2+\left(a_1b_2-a_2b_1\right)^2}}\right|$$ **[1]**

$$\text{S.D.}=\left|\frac{\begin{vmatrix} -6 & -15 & 3 \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix}}{\sqrt{(-4-2)^2+(-3-12)^2+(6-3)^2}}\right|$$ **[1]**

$$=\frac{270}{\sqrt{270}}=\sqrt{270}\;=3\sqrt{30}\text{ units}$$ **[1]**

Alternatively :

$\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k})$

$\vec{r} = (-3\hat{i} - 7\hat{j} + 6\hat{k}) + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k})$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix} = -6\hat{i} - 15\hat{j} + 3\hat{k}$$ [½]

$\vec{a}_2 - \vec{a}_1 = -6\hat{i} - 15\hat{j} + 3\hat{k}$ [½]

$$\text{S.D.} = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$ [½]

$$\left| \frac{(-6\hat{i} - 15\hat{j} + 3\hat{k}) \cdot (-6\hat{i} - 15\hat{j} + 3\hat{k})}{|-6\hat{i} - 15\hat{j} + 3\hat{k}|} \right| = \frac{36 + 225 + 9}{\sqrt{36 + 225 + 9}} = \frac{270}{\sqrt{270}}$$ [1]

$= \sqrt{270} = 3\sqrt{30}$ units [½]

खण्ड–C / SECTION–C

11. Let $I = \int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$ ......(1)

$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)} dx$ $\left[\because \int_0^a f(x)dx = \int_0^a f(a - x)dx \right]$

$\Rightarrow I = \int_0^{\pi} \frac{\pi - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$ ......(2) [½]

(1) + (2) gives :

$$2I = \int_0^{\pi} \frac{\pi}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$2I = 2\pi \int_0^{\pi/2} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$ $\left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ If } f(2a - x) = f(x) \right]$ [1]

On dividing numerator and denominator by $\cos^2 x$, we get

$$I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx$$

Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

when $x = 0$, then $t = 0$

when $x = \frac{\pi}{2}$, then $t = \infty$ [½]

$$\therefore I = \pi \int_0^{\infty} \frac{1}{a^2 + b^2 t^2} dt$$

$\Rightarrow \quad I = \frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\right]_0^{\infty}$ **[1]**

$I = \frac{\pi}{ab}[\tan^{-1}\infty - \tan^{-1}0]$

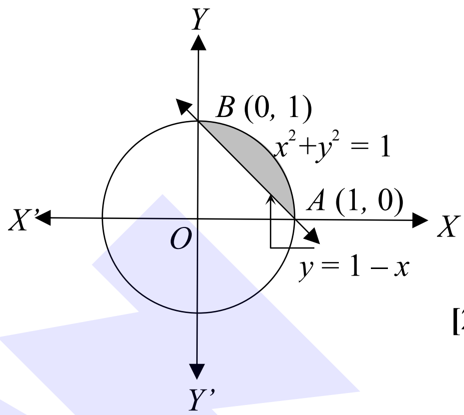
$I = \frac{\pi}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{2ab}$ **[1]**

12. $x^2 + y^2 = 1, x + y = 1$ meet when

$x^2 + (1-x)^2 = 1 \Rightarrow x^2 + 1 + x^2 - 2x = 1$

$\Rightarrow 2x^2 - 2x = 0 \Rightarrow 2x(x-1) = 0$

$\Rightarrow x = 0, x = 1 \quad \Rightarrow y = 1, y = 0$, *i.e.*, A(1,0); B(0,1)



Required area $= \int_0^1 [\sqrt{1-x^2} - (1-x)]\, dx$ **[2]**

$= \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x - x + \frac{x^2}{2}\right]_0^1$

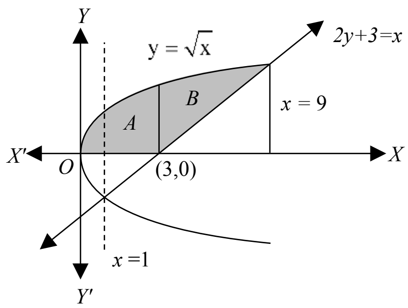
$= \frac{1}{2}.\frac{\pi}{2} - 1 + \frac{1}{2} = \left(\frac{\pi}{4} - \frac{1}{2}\right)$ sq. units. **[2]**

OR

Solving $y^2 = x$ and $x = 2y + 3$

$4y^2 = (x-3)^2,\ 4x = x^2 - 6x + 9$

$\Rightarrow x^2 - 10x + 9 = 0 \Rightarrow (x-1)(x-9) = 0 \Rightarrow x = 1, 9$ **[½]**



Required area = $A+B = \int_0^3 \sqrt{x}dx + \int_3^9 \left[\sqrt{x} - \left(\frac{x-3}{2}\right)\right]dx$

$= \frac{2}{3}[x^{3/2}]_0^3 + \frac{2}{3}[x^{3/2}]_3^9 - \frac{1}{2}\left[\frac{x^2}{2} - 3x\right]_3^9$ **[1½]**

$= \frac{2}{3}(3\sqrt{3}) + \frac{2}{3}[9\times 3 - 3\sqrt{3}] - \frac{1}{2}\left[\left(\frac{81}{2} - 27\right) - \left(\frac{9}{2} - 9\right)\right]$ **[1]**

$= 18 - \frac{1}{2}[36 - 18] = 18 - 9 = 9$ *sq. units.* **[1]**

13. Equation of plane passing through the line of intersection of the planes $3x - y - 4z = 0$ and $x + 3y + 6 = 0$ is,

$(3x - y - 4z) + \lambda(x + 3y + 6) = 0 \qquad ......(i)$ **[½]**

Given, distance of plane (i) from origin is 1.

$\therefore \quad \left|\frac{6\lambda}{\sqrt{(3+\lambda)^2 + (3\lambda-1)^2 + 4^2}}\right| = 1$ **[½]** or $36\lambda^2 = 10\lambda^2 + 26$ or $\lambda = \pm 1$ **[1]**

Put the value of λ in (i),

$\therefore (3x - y - 4z) \pm (x + 3y + 6) = 0$

or $4x + 2y - 4z + 6 = 0$ or $2x + y - 2z + 3 = 0$

and $2x - 4y - 4z - 6 = 0$

Thus the required planes are $x - 2y - 2z - 3 = 0$ and $2x + y - 2z + 3 = 0$. **[2]**

CASE-BASED/DATA-BASED

14. (i) $P(E_1) = \frac{25}{100}$, $P(E_2) = \frac{35}{100}$, $P(E_3) = \frac{40}{100}$

$P(F/E_1) = \frac{5}{100}$, $P(F/E_2) = \frac{4}{100}$, $P(F/E_3) = \frac{2}{100}$ **[½]**

$\Rightarrow$ Required probability $= 1 - P\left(\frac{E_3}{F}\right)$

$$= 1 - \left[\frac{P(E_3).P(F/E_3)}{P(E_1).P(F/E_1) + P(E_2).P(F/E_2) + P(E_3).P(F/E_3)}\right]$$ **[½]**

$$= 1 - \left[\frac{\frac{40}{100} \times \frac{2}{100}}{\left(\frac{25}{100} \times \frac{5}{100}\right) + \left(\frac{35}{100} \times \frac{4}{100}\right) + \left(\frac{40}{100} \times \frac{2}{100}\right)}\right]$$

$$= 1 - \left(\frac{80}{125 + 140 + 80}\right) = \frac{53}{69}$$ **[1]**

(ii) $\sum_{i=1}^{3} P(E_i / F) = P(E_1/F) + P(E_2/F) + P(E_3/F)$ **[½]**

$$= \frac{P(E_1) \times P(F/E_1)}{\sum_{i=1}^{3} P(E_i).P(F/E_i)} + \frac{P(E_2) \times P(F/E_2)}{\sum_{i=1}^{3} P(E_i).P(F/E_i)} + \frac{P(E_3) \times P(F/E_3)}{\sum_{i=1}^{3} P(E_i).P(F/E_i)}$$

$$= \frac{\sum_{i=1}^{3} P(E_i).P(F/E_i)}{\sum_{i=1}^{3} P(E_i).P(F/E_i)}$$ **[1]**

$= 1$ **[½]**