Board of Secondary Education Rajasthan, Ajmer

SUBJECT: MATHEMATICS
CLASS-XII
SOLUTIONS

(SECTION-A)

1.

(i) **(B)**

gof(x) = g[f(x)]

$\therefore$ g[f(1)] = g(2) = 3

g[f(3)] = g(5) = 1

and g[f(4)] = g(1) = 3

Hence ; range of gof = {1, 3}

(ii) **(A)**

$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$= \sin^{-1}\left[\sin\left(\pi - \frac{\pi}{3}\right)\right] \quad \because \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$= \frac{\pi}{3}$

(iii) **(C)**

Given $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$; $A^2 = I$

Now $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & a^2 + \beta\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \alpha^2 + \beta\gamma = 1$ or $1 - \alpha^2 - \beta\gamma = 0$

(iv) **(A)**

Given, $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = 0$

Expanding along R_1

$(1 + a)\,[(1 + a)^2 - (1)] - 1((1 + a) - 1) + 1(1 - 1 - a) = 0$

$(1 + a)\,(1 + a^2 + 2a - 1) - a - a = 0$

$a^2 + 2a + a^3 + 2a^2 - 2a = 0$

$a^3 + 3a^2 = 0$

$a^2(a + 3) = 0$

$\Rightarrow a = 0$ or $a = -3$.

(v) (A)

$y = \sqrt{\sin x + y}$

$y^2 = \sin x + y$ [On squaring both side]

$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$

$\frac{dy}{dx}(2y-1) = \cos x$

$\boxed{\frac{dy}{dx} = \frac{\cos x}{2y-1}}$

(vi) (B)

$\int_0^a \frac{1}{9x^2+1}dx = \frac{\pi}{12}$

$\int_0^a \frac{dx}{(3x)^2+1} = \frac{\pi}{12}$

$\frac{1}{3}\left[\tan^{-1}(3x)\right]_0^a = \frac{\pi}{12}$

$\frac{1}{3}\left[\tan^{-1}3a - \tan^{-1}0\right] = \frac{\pi}{12}$

$\tan^{-1}3a = \frac{\pi}{4}$

$3a = \tan\frac{\pi}{4}$

$3a = 1$

$\boxed{a = \frac{1}{3}}$

(vii) (C)

Given $\cos x\frac{dy}{dx} + y\sin x = 1$

$\Rightarrow \frac{dy}{dx} + y\tan x = \sec x$

I.F. = $e^{\int \tan x dx}$

$= e^{\log\sec x} = \sec x$

(viii) (A)

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\Rightarrow \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

(ix) **(B)**

Required probability of two hits

$= P(A)\,P(B)\,P(C') + P(A)\,P(B')\,P(C) + P(A')\,P(B)\,P(C)$

$= (0.4 \times 0.3 \times 0.8) + (0.4 \times 0.7 \times 0.2) + (0.6 \times 0.3 \times 0.2)$

$= 0.096 + 0.056 + 0.036$

$= 0.188$

(x) **(A)**

Given, $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$A^2 = A \times A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{......... (1)}$$

Also $$A^3 = A^2 \times A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \quad \text{......... (2)}$$

From (1) & (2) we can conclude that $A^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$

(xi) **(A)**

Let $z = \cos^{-1}(2x^2 - 1)$

Put $x = \cos\theta$ ......(1)

$z = \cos^{-1}(2\cos^2\theta - 1)$

$z = \cos^{-1}(\cos 2\theta)$

$z = 2\theta$

$z = 2\cos^{-1} x$

$$\frac{dz}{dx} = -\frac{2}{\sqrt{1-x^2}} \quad \text{....(1)}$$

Let $y = \cos^{-1} x$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \quad \text{......(2)}$$

$$\frac{dz}{dy} = \frac{\frac{dz}{dx}}{\frac{dy}{dx}} = \frac{\frac{-2}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}} = 2$$

$$\boxed{\frac{dz}{dy} = 2}$$

(xii) **(C)**

Let θ be the angle between $\vec{a}$ and $\vec{b}$

$$\Rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

$$= \frac{(2\hat{i} + \hat{j} - 3\hat{k}) \cdot (3\hat{i} - 2\hat{j} - 2\hat{k})}{\sqrt{14}\quad\sqrt{17}}$$

$$= \frac{10}{\sqrt{238}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{10}{\sqrt{238}}\right)$$

2.

(i) Given ; f and g are two invertible functions such that gof is defined.

$\Rightarrow$ $(gof)^{-1} = f^{-1} \text{ o } g^{-1}$

(ii) $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$

$\tan^{-1}\left[\frac{x+y}{1-xy}\right] = \frac{\pi}{4}$

$\frac{x+y}{1-xy} = 1 \Rightarrow x + y = 1 - xy \Rightarrow \boxed{x + y + xy = 1}$

(iii) $A^T = A$

(iv) $\sqrt{x} + \sqrt{y} = 1$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$

$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$

at $\left(\frac{1}{4}, \frac{1}{4}\right)$

$\frac{dy}{dx} = -1$

(v) $\int_0^a \frac{1}{1+4x^2}dx = \frac{\pi}{8}$

$\int_0^a \frac{1}{1+(2x)^2}dx = \frac{\pi}{8}$

$\frac{1}{2}\left[\tan^{-1}(2x)\right]_0^a = \frac{\pi}{8}$

$\frac{1}{2}\left[\tan^{-1}2a - \tan^{-1}0\right] = \frac{\pi}{8}$

$\tan^{-1}2a = \frac{\pi}{4}$

$2a = \tan\frac{\pi}{4}$

$2a = 1$

$\boxed{a = \frac{1}{2}}$

(vi) Let $\vec{a} = 6\hat{i} + 2\hat{j} - 3\hat{k}$

$\Rightarrow \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{6\hat{i}+2\hat{j}-3\hat{k}}{\sqrt{36+4+9}}$

$= \frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k}$

Hence; required direction cosines are $\frac{6}{7}, \frac{2}{7}$ and $\frac{-3}{7}$

node06\B0BA-BB\Kota\Board Material\Mathematics\Model paper (RBSE)

3.

(i) Given ; $f : R \to R,\ g : R \to R$

and $f(x) = 2x - 3$,

$g(x) = x^3 + 5$

Since f and g are bijective functions, we can calculate f^{-1} and g^{-1}

Now ; $y = f(x) = 2x - 3$

$$\Rightarrow x = \frac{y+3}{2}$$

$$\text{or } \quad f^{-1}(y) = \frac{y+3}{2} \Rightarrow f^{-1}(x) = \frac{x+3}{2}$$

Also ; $y = g(x) = x^3 + 5$

$x = (y-5)^{1/3}$

or $g^{-1}(y) = (y-5)^{1/3}$

$\Rightarrow g^{-1}(x) = (x-5)^{1/3}$

$$\therefore \ (fog)^{-1}(x) = g^{-1}of^{-1}(x)$$

$$= g^{-1}[f^{-1}(x)]$$

$$= g^{-1}\left[\frac{x+3}{2}\right]$$

$$= \left(\frac{x+3}{2} - 5\right)^{1/3} = \left(\frac{x-7}{2}\right)^{1/3}$$

(ii) $\cos^{-1}\left(\cos\frac{5\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$$= \cos^{-1}\cos\left(\pi + \frac{2\pi}{3}\right) + \sin^{-1}\sin\left(\pi - \frac{\pi}{3}\right)$$

$$= \pi - \frac{2\pi}{3} + \frac{\pi}{3}$$

$$= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

(iii) Given $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix}$$

Also $\quad A^2 = \lambda A$

$$\begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix} = \lambda\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$\therefore 3\lambda = 18$ or $\lambda = 6$

(iv) Given $|A| = 5$

$|B| = 3$

$|3AB| = 27|A|\,|B|$ (by using properties $|kA| = k^n|A|$

$= 27 \times 5 \times 3$ where $n \times n$ is order of matrix

$= 27 \times 15$

$= 405$

(v) $y = \tan^{-1}x$

$\tan y = x$

Diff. w.r.t. x

$\sec^2 y \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \cos^2 y$

$\dfrac{d^2y}{dx^2} = -2\cos y \sin y \cdot \dfrac{dy}{dx}$

$= -\sin 2y \cos^2 y$

(vi) $\displaystyle\int \frac{\cos x - \sin x}{\sqrt{1+\sin 2x}}dx$

$\displaystyle= \int \frac{\cos x - \sin x}{(\cos x + \sin x)^2}dx$

Put $\cos x + \sin x = t$

$(-\sin x + \cos x)dx = dt$

$\displaystyle= \int \frac{\cos x - \sin x}{(\cos x + \sin x)}dx$

$\displaystyle= \int \frac{dt}{t}$

$= \log (t) + C$

$= \log(\cos x + \sin x) + C$

(vii) given $x\sqrt{x+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0$

By variable separating

$\dfrac{y}{\sqrt{1+y^2}}dy = -\dfrac{x}{\sqrt{1+x^2}}dx$

Integrating both side

$\dfrac{y}{\sqrt{1+y^2}}dx = -\dfrac{x}{\sqrt{1+x^2}}dx$

Put $1 + y^2 = t$ & $1 + x^2 = z$

$2ydy = dt$ & $2xdx = dz$

$\displaystyle\frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\int \frac{dz}{\sqrt{z}} \Rightarrow \sqrt{t} = \sqrt{z}$

$\Rightarrow \sqrt{1+y^2} = \sqrt{1+x^2} + C$

(viii) Given ; adjacent sides of a parallelogram are $\vec{a}$ and $\vec{b}$

$$\Rightarrow \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$$

$$= 20\hat{i} - 5\hat{j} - 5\hat{k}$$

$$\therefore \ |\vec{a}\times\vec{b}| = \sqrt{400+25+25} = \sqrt{450}$$

Hence ; area of parallelogram = $|\vec{a}\times\vec{b}|$

$$= \sqrt{450} = 15\sqrt{2} \text{ sq.units}$$

(ix) P(second ball is blue)

= P(first ball red, second ball blue) + P(first ball blue, second ball also blue)

$$= \left(\frac{4}{9}\times\frac{5}{8}\right) + \left(\frac{5}{9}\times\frac{4}{8}\right)$$

$$= \frac{20}{72} + \frac{20}{72} = \frac{40}{72} = \frac{5}{9}$$

(x) $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$

$$A^{-1} = \frac{\text{adj}(A)}{|A|}, |A| \neq 0$$

$|A| = (4 \times 2) - (3 \times -1)$

$= 8 + 3 = 11$

$|A| \neq 0$

$$\text{adj }(A) = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

(xi) Given $x + y = \tan^{-1}y$

differentiation w.r.t. x

$$1 + y' = \frac{1}{1+y^2}y'$$

$$(1 + y')(1 + y^2) = y'$$

$$1 + y^2 + y' + y^2y' = y'$$

$$\Rightarrow \quad y^2y' + y^2 + 1 = 0$$

(xii) Given ; $\vec{a}$ and $\vec{b}$ are parallel.

$$\Rightarrow \frac{5}{3} = \frac{\lambda}{9} \quad \text{or} \quad \lambda = 15$$

(SECTION-B)

4. $f(x) = \dfrac{4x+3}{3x+4}$

(i) For one-one function : we have $f(x_1) = f(x_2)\ \forall\ x_1, x_2\ \&\ R - \left\{-\dfrac{4}{3}\right\}$

$$\Rightarrow \quad \frac{4x_1+3}{3x_1+4} = \frac{4x_2+3}{3x_2+4}$$

$$\Rightarrow \quad 12x_1x_2 + 16x_1 + 9x_2 + 12$$

$$= 12x_1x_2 + 9x_1 + 16x_2 + 12$$

$$\Rightarrow \quad 7x_1 - 7x_2 = 0 \qquad \Rightarrow \qquad x_1 = x_2$$

function is one-one.

(ii) For onto function : Let $y = f(x)$

$$y = \frac{4x+3}{3x+4}$$

$$\Rightarrow \quad 3xy + 4y = 4x + 3$$

$$\Rightarrow \quad x(3y-4) = 3 - 4y$$

$$\Rightarrow \quad x = \frac{3-4y}{3y-4} \in R - \left\{-\frac{4}{3}\right\}, \forall y \in R - \left\{\frac{4}{3}\right\}$$

if $\quad 3y - 4 \neq 0$

$$y \neq \frac{4}{3}$$

every element of codomain has preimages in domain.

So range = codomain

function is onto

Now, $f(x)$ is bijective

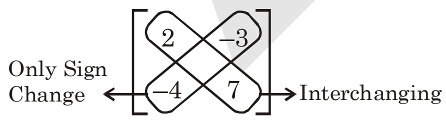
since it it one-one and onto both.

5. Given, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$|A| = 14 - (12) = 2 \neq 0$

Hence, A is invertible.

$$A^{-1} = \frac{\text{adj.}(A)}{|A|}$$



$$\text{adj}A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

Hence, $A^{-1} = \dfrac{\text{adj.}(A)}{|A|}$

$$= \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{2} & \frac{3}{2} \\ 2 & 1 \end{bmatrix}$$

Now RHS = 9I – A

$$= 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} \quad = 2A^{-1} = \text{LHS } = 2\begin{bmatrix} \frac{7}{2} & \frac{3}{2} \\ 2 & 1 \end{bmatrix}$$

Hence proved

6. $\text{LHS} = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2(a-1) & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= (a-1)^2 \begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Expanding

$(a – 1)^2.(a – 1) = (a – 1)^3$.

7. $\frac{dy}{dx} = \cos(\sin x).\cos x$ ...(2)

$$\frac{d^2y}{dx^2} = \cos(\sin x)\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}\cos(\sin x)$$

$$\frac{d^2y}{dx^2} = -\sin x\cos(\sin x) + \cos x.\{-\sin(\sin x)\}.\cos x \quad ...(3)$$

Put sin(sinx) = y from (1) and

$\cos(\sin x) = \frac{1}{\cos x}\left(\frac{dy}{dx}\right)$ from (2) in (3)

$$\frac{d^2y}{dx^2} = \frac{-\sin x}{\cos x}.\frac{dy}{dx} - y\cos^2 x$$

$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0$$

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8. $\int \frac{x-5}{(x-3)^3} e^x dx$

$= \int e^x \left[\frac{(x-3)-2}{(x-3)^3} \right] dx$

$= \int e^x \left[\frac{(x-3)}{(x-3)^3} - \frac{2}{(x-3)^3} \right] dx$

$= \int e^x \left[\frac{1}{(x-3)^2} + \left\{ \frac{-2}{(x-3)^3} \right\} \right] dx$

$= e^x / (x-3)^2 + C$ $\qquad \left[\because \int e^x [f(x) + f'(x)] dx = e^x f(x) + C \right]$

9. Let E_1 : select first purse and E_2 : select second purse

A : silver coin is drawn

$\Rightarrow \ P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2}, P(A/E_1) = \frac{3}{9}, P(A/E_2) = \frac{4}{7}$

$\therefore \ P(A) = P(E_1)\, P(A/E_1) + P(E_2).P(A/E_2)$ (by total probability theorem)

$= \frac{1}{2} \times \frac{3}{9} + \frac{1}{2} \times \frac{4}{7}$

or $P(A) = \frac{1}{6} + \frac{2}{7} = \frac{19}{42}$

10. Let $D = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$\therefore \quad A + B - D = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 1-1-a & 3-2-b \\ 3+0-c & 2+5-d \\ 2+3-e & 5+1-f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow -a = 0 \Rightarrow a = 0, \quad 1 - b = 0 \Rightarrow \quad b = 1,$

$3 - c = 0 \Rightarrow \quad c = 3, \quad 7 - d = 0 \Rightarrow \quad d = 7,$

$5 - e = 0 \Rightarrow \quad e = 5, \quad 6 - f = 0 \Rightarrow \quad f = 6$

$\therefore \ D = \begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$

11. $x = a(\cos 2t + 2t \sin 2t)$

$\frac{dx}{dt} = a[-2\sin 2t + 2(2t\cos 2t + \sin 2t)]$

$\frac{dx}{dt} = 4at\cos 2t$ ....(1)

$y = a(\sin 2t - 2t\cos 2t)$

$$\frac{dy}{dt} = a\left[2\cos 2t - 2\left(-2t\sin 2t + \cos 2t\right)\right]$$

$$\frac{dy}{dt} = 4at\sin 2t \qquad ....(2)$$

By chain rule

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at\sin 2t}{4at\cos 2t}$$

$$\frac{dy}{dx} = \tan 2t$$

$$\frac{d^2y}{dx^2} = 2\sec^2 2t \times \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = 2\sec^2 2t \times \frac{1}{4at\cos 2t}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2at} \times \sec^3 2t$$

12. Matrix from of system of linear equation $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{bmatrix}$

System of linear equation's has unique solution.

If $|A| \neq 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$$

$\Rightarrow \quad 1(k + 2) - 1(2k + 3) + 1(4 - 3) \neq 0$

$\Rightarrow \quad k \neq 0$

13. $I = \displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad ....(1)$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \qquad ....(2) \qquad \left[\text{Using } \int_a^b f(x) = \int_a^b f(a+b-x)\right]$$

adding equations (1) & (2)

$$2I = \int_{\pi/6}^{\pi/3} dx \;\Rightarrow\; 2I = \frac{\pi}{6} \;\Rightarrow\; I = \frac{\pi}{12}$$

14. $\log\left(\frac{dy}{dx}\right) = 3x + 4y$

$\Rightarrow \quad \frac{dy}{dx} = e^{3x+4y} \qquad \Rightarrow \quad \frac{dy}{dx} = e^{3x} \cdot e^{4y}$

$\Rightarrow \quad \int e^{-4y}\, dy = \int e^{3x}\, dx \qquad \Rightarrow \quad \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C \qquad ...(i)$

Given that y = 0 when x = 0

$C = \frac{-1}{4} - \frac{1}{3} = \frac{-7}{12}$

$\Rightarrow \quad \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} - \frac{7}{12}$ (from Equation (i))

or $\quad 3e^{-4y} + 4e^{3x} = 7$

15. $\vec{s} = (2\hat{i} + 4\hat{j} - 5\hat{k}) + (\lambda\hat{i} + 2\hat{j} + 3\hat{k})$

$\vec{s} = \hat{i}(2+\lambda) + 6\hat{j} - 2\hat{k}$

Unit vector of $\vec{s} = \hat{s} = \frac{\vec{s}}{|\vec{s}|} = \frac{(\lambda+2)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(\lambda+2)^2 + 6^2 + 2^2}}$

Given : $(\hat{i} + \hat{j} + \hat{k}).\hat{s} = 1$

$\therefore \quad \frac{(\lambda+2) + 6 - 2}{\sqrt{(\lambda+2)^2 + 40}} = 1$

or $\quad (\lambda + 6) = \sqrt{(\lambda+2)^2 + 40}$ (on squaring both sides)

$\lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44$

$\Rightarrow 8\lambda = 8 \quad \Rightarrow \lambda = 1$

16. Let E_1 : students has 100% attendence

E_2 : irregular students

E : students has A grade

$P(E_1) = \frac{30}{100}$ and $P(E_2) = \frac{70}{100}$

$P(E/E_1) = \frac{70}{100}; \; P(E/E_2) = \frac{10}{100}$

Req. prob. $= P(E_1/E) = \frac{P(E_1) \times P(E/E_1)}{P(E_1) \times P(E/E_1) + P(E_2) \times P(E/E_2)}$ (using Bayes' theorem)

$$= \frac{\frac{30}{100} \times \frac{70}{100}}{\left(\frac{30}{100} \times \frac{70}{100}\right) + \left(\frac{70}{100} \times \frac{10}{100}\right)}$$

$$= \frac{2100}{2100 + 700} = \frac{21}{28} = \frac{3}{4}$$

(SECTION-C)

17. $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ \frac{-1}{\sqrt{2}}\le x\le 1$

In LHS

put x = cos2θ

$$\tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right]$$

$$=\tan^{-1}\left[\frac{\sqrt{1+2\cos^2\theta-1}-\sqrt{1-1+2\sin^2\theta}}{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-1+2\sin^2\theta}}\right]$$

$$=\tan^{-1}\left[\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right]$$

$$=\tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$

$$=\tan^{-1}\left[\frac{\tan(\pi/4)-\tan\theta}{1+\tan(\pi/4).\tan\theta}\right]$$

$$=\tan^{-1}[\tan(\pi/4-\theta)]$$

$$=\frac{\pi}{4}-\theta \qquad \text{as}\left\{\begin{aligned} x&=\cos 2\theta\\ \text{so, }\theta&=\frac{\cos^{-1}x}{2}\end{aligned}\right\}$$

$$=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x = \text{RHS} \qquad \text{proved}$$

OR

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4} \qquad ...(i)$$

Use formula, $\tan^{-1}\left[\frac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\left(\frac{x-2}{x-4}\right)\left(\frac{x+2}{x+4}\right)}\right]=\frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left[\frac{(x-2)(x+4)+(x+2).(x-4)}{(x-4).(x+4)-(x-2).(x+2)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{(x-2)(x+4)+(x+2).(x-4)}{(x-4).(x+4)-(x-2).(x+2)}=1$$

$$\Rightarrow \quad \frac{x^2-8-2x+x^2-8+2x}{x^2-16-x^2+4}=1$$

$$\Rightarrow \quad \frac{2x^2-16}{-12}=1$$

$$\Rightarrow \quad 2x^2 = -12 + 16 = 4$$

$$\Rightarrow \quad x^2 = 2 \quad \Rightarrow x=\pm\sqrt{2}$$

18. Given $(x^2 + y^2)^2 = xy$,

$x^4 + y^4 + 2x^2y^2 = xy$

Diff. at both side w.r.t. x

$$4x^3 + 4y^3\frac{dy}{dx} + 2\left[2xy^2 + x^2.2y\frac{dy}{dx}\right] = x\frac{dy}{dx} + y$$

$$\frac{dy}{dx}(4y^3 + 4x^2y - x) = (y - 4x^3 - 4xy^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 4x^3 - 4xy^2}{4y^3 + 4x^2y - x}$$

OR

Function is parametric therefore $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ ....(1)

Now, $x = a(2\theta - \sin 2\theta)$

Differentiation w.r.t. θ

$$\frac{dx}{d\theta} = a(2 - 2\cos 2\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = 2a(1 - \cos 2\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = 4a\sin^2\theta \quad ......(2)$$

$y = a(1 - \cos 2\theta)$

Differentiation w.r.t. θ

$$\frac{dy}{d\theta} = 2a\sin 2\theta$$

$$\Rightarrow \frac{dy}{d\theta} = 4a\sin\theta\cos\theta \quad ....(3)$$

Put the value of equation (2) and (3) in (1)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$\frac{dy}{dx} = \frac{4a\sin\theta\cos\theta}{4a\sin^2\theta}$$

$$\frac{dy}{dx} = \frac{\cos\theta}{\sin\theta}$$

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{3}} = \frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\sqrt{3}}$$

19. $I=\int e^{2x}\frac{(2x-5)}{(2x-3)^3}dx$

$I=\int e^{2x}\left[\frac{(2x-3)}{(2x-3)^3}-\frac{2}{(2x-3)^3}\right]dx$

$I=\int e^{2x}\left[\frac{1}{(2x-3)^2}-\frac{2}{(2x-3)^3}\right]dx$

Put 2x = t

$dx=\frac{dt}{2}$

$I=\frac{1}{2}\int e^{t}\left[\frac{1}{(t-3)^2}-\frac{2}{(t-3)^3}\right]dt$

$I=\frac{1}{2}\frac{e^t}{(t-3)^2}+C$ $\left[\because \int e^x\left[f(x)+f'(x)\right]dx=e^x f(x)+C\right]$

$I=\frac{1}{2}\frac{e^{2x}}{(2x-3)^2}+C$

OR

$I=\int\frac{x^2+x+1}{(x^2+1)(x+2)}dx$

Let $\frac{x^2+x+1}{(x+2)(x^2+1)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+1}$ .... (1)

$x^2+x+1=A(x^2+1)+(Bx+C)(x+2)$

At $x=-2$ $A=\frac{3}{5}$

and at $x=0$ $A+2C=1 \Rightarrow C=\frac{1}{5}$

Equating the coeff. of x^2

$1=A+B \Rightarrow B=1-\frac{3}{5}=\frac{2}{5}$

Put in equation (1) and integrate

$\therefore I=\int\frac{x^2+x+1}{(x+2)(x^2+1)}dx=\frac{3}{5}\int\frac{1}{x+2}dx+\int\frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1}dx$

$=\frac{3}{5}\int\frac{1}{x+2}dx+\frac{1}{5}\int\frac{2x}{x^2+1}dx+\frac{1}{5}\int\frac{1}{x^2+1}dx$

$=\frac{3}{5}\log(x+2)+\frac{1}{5}\log(x^2+1)+\frac{1}{5}\tan^{-1}x+C$

20. Let α be the angle made by vector with coordinate axes

Then $\vec{r} = 3\sqrt{3}(\ell\hat{i} + m\hat{j} + n\hat{k})$ $\quad (\because |\vec{r}| = 3\sqrt{3})$

where $\ell = \cos\alpha$, $m = \cos\alpha$, $n = \cos\alpha$

$\because \ell^2 + m^2 + n^2 = 1$

$\Rightarrow \quad 3.\cos^2\alpha = 1 \qquad \Rightarrow \quad \cos\alpha = \pm\dfrac{1}{\sqrt{3}}$

$\therefore \quad \vec{r} = 3\sqrt{3}\left(\pm\dfrac{\hat{i}}{\sqrt{3}} \pm \dfrac{\hat{j}}{\sqrt{3}} \pm \dfrac{\hat{k}}{\sqrt{3}}\right)$

or $\quad \vec{r} = \pm 3(\hat{i} + \hat{j} + \hat{k})$

OR

Given $\vec{a}$ & $\vec{b}$ are unit vector i.e. $|\vec{a}| = 1 = |\vec{b}|$

and $|\sqrt{3}\vec{a} - \vec{b}| = 1 \quad \Rightarrow \quad |\sqrt{3}\vec{a} - \vec{b}|^2 = 1$

$\Rightarrow \quad (\sqrt{3}\vec{a} - \vec{b}).(\sqrt{3}\vec{a} - \vec{b}) = 1$

or $\quad 3|\vec{a}|^2 - \sqrt{3}\vec{a}.\vec{b} - \sqrt{3}\vec{b}.\vec{a} + |\vec{b}|^2 = 1 \qquad (\because \vec{a}.\vec{a} = |\vec{a}|^2)$

$\Rightarrow \quad 3 - 2\sqrt{3}\vec{a}.\vec{b} + 1 = 1 \qquad (\because \vec{a}.\vec{b} = \vec{b}.\vec{a})$

or $\quad \vec{a}.\vec{b} = \dfrac{\sqrt{3}}{2} \Rightarrow |\vec{a}||\vec{b}|\cos\theta = \dfrac{\sqrt{3}}{2} \qquad (\because \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta)$

$\Rightarrow \cos\theta = \dfrac{\sqrt{3}}{2} \quad \Rightarrow \theta = \dfrac{\pi}{6}$

(SECTION-D)

21. $I = \displaystyle\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$

$I = \displaystyle\int_0^{\pi} \frac{(\pi - x)dx}{a^2\cos^2 x + b^2\sin^2 x} \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$2I = \displaystyle\int_0^{\pi} \frac{\pi dx}{a^2\cos^2 x + b^2\sin^2 x}$

$I = \dfrac{\pi}{2}\displaystyle\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$I = \dfrac{\pi}{2} \times 2\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} \qquad \left[\because \int_0^a f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x)\right]$

$$I = \pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x dx}{a^2 + b^2 \tan^2 x}$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

when $x = 0 \Rightarrow t = 0$ & $x = \pi/2 \Rightarrow t = \infty$

$$I = \pi\int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} = \frac{\pi}{b^2}\int_0^{\infty} \frac{dt}{t^2 + \left(\frac{a}{b}\right)^2}$$

$$I = \left(\frac{\pi}{b^2}\right)\frac{1}{\left(\frac{a}{b}\right)}\left|\tan^{-1}\left(\frac{t}{a/b}\right)\right|_0^{\infty} = \frac{\pi}{ab}.\frac{\pi}{2} = \frac{\pi^2}{2ab}.$$

OR

$$\int \frac{dx}{\cos^4 x + \sin^4 x}$$

$$= \int \frac{\frac{1}{\cos^4 x}}{1 + \tan^4 x} dx$$

$$= \int \frac{\sec^2 x . \sec^2 x}{1 + \tan^4 x} dx$$

$$= \int \frac{(1 + \tan^2 x)\sec^2 x}{1 + \tan^4 x} dx$$

put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$= \int \frac{(1 + t^2)dt}{1 + t^4}$$

$$= \int \frac{\left(\frac{1}{t^2} + 1\right)dt}{\frac{1}{t^2} + t^2}$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right)dt}{\left(t - \frac{1}{t}\right)^2 + 2}$$

put $t - \frac{1}{t} = z \Rightarrow \left(1 + \frac{1}{t^2}\right)dt = dz$

$$= \int \frac{dz}{z^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\left(t - \frac{1}{t}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{1}{\sqrt{2}} \left(\tan x - \frac{1}{\tan x} \right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - \cot x}{\sqrt{2}} \right) + C$$

22. $x\frac{dy}{dx} = y - x\tan(y/x)$

$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{y}{x}\right) - \tan\left(\frac{y}{x}\right)$

The given equation is homogenous differential equation.

Put $\frac{y}{x} = v$ i.e. $y = vx \Rightarrow dy/dx = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} = v - \tan v$

$\Rightarrow \quad x\frac{dv}{dx} = -\tan v$

$\Rightarrow \quad \frac{dv}{\tan v} = -\frac{dx}{x}$

Integrating both sides; we get :

$\int \frac{dv}{\tan v} = \int \frac{-dx}{x}$

$\Rightarrow \quad \int \cot v \, dv = -\int dx/x$

$\Rightarrow \quad \log|\sin v| = -\log|x| + \log C$

$\Rightarrow \quad \log|\sin v| = \log\left|\frac{C}{x}\right|$

$\Rightarrow \quad \sin\left(\frac{y}{x}\right) = C/x$

or $\quad x\sin\left(\frac{y}{x}\right) = C$

OR

$\frac{dy}{dx} = -\left[\frac{x + y\cos x}{1 + \sin x}\right]$

$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{\cos x}{1 + \sin x}\right)y = \frac{-x}{1 + \sin x}$ .....(i)

Eq. (i) is a linear diff. eq. of the form $\frac{dy}{dx} + Py = Q$;

where, $P = \left(\frac{\cos x}{1 + \sin x}\right)$ and $Q = \left(\frac{-x}{1 + \sin x}\right)$

$\Rightarrow \quad$ I.F. $= e^{\int P\,dx} = e^{\int \frac{\cos x}{1+\sin x}dx} = e^{\log|1+\sin x|} = 1 + \sin x$

Hence, the solution is

$y \times \text{I.F.} = \int Q \times \text{I.F.}dx + C$

$\Rightarrow \quad y \times (1 + \sin x) = \int \left(\frac{-x}{1+\sin x}\right) \times (1+\sin x)\,dx + C$

$\Rightarrow \quad y(1 + \sin x) = \int -x\,dx + C$

or $\quad y(1+\sin x) = \frac{-x^2}{2} + C$

23. Let E_1 = event of drawing bag X

E_2 = event of drawing bag Y

E = event of drawing one white and one black ball.

$\Rightarrow \quad P(E_1) = P(E_2) = \frac{1}{2}$

Also, $P\left(\frac{E}{E_1}\right) = \left(\frac{4}{6}\right)\left(\frac{2}{5}\right) + \left(\frac{2}{6}\right)\left(\frac{4}{5}\right) = \frac{16}{30}$

$P\left(\frac{E}{E_2}\right) = \left(\frac{3}{6}\right)\left(\frac{3}{5}\right) + \left(\frac{3}{6}\right)\left(\frac{3}{5}\right) = \frac{18}{30}$

Req. prob. $= P(E_2/E) = \frac{P(E_2) \times P(E/E_2)}{P(E_1) \times P(E/E_1) + P(E_2) \times P(E/E_2)}$ (using Baye's Th.)

$$= \frac{\frac{1}{2} \times \frac{18}{30}}{\left(\frac{1}{2} \times \frac{16}{30}\right) + \left(\frac{1}{2} \times \frac{18}{30}\right)} = \frac{9}{17}$$

OR

Let A_i and B_i be the events of throwing 10 by A and B resp.

$P(A_i) = P(B_i) = \frac{3}{36} = \frac{1}{12}$

$P(\bar{A}_i) = P(\bar{B}_i) = \frac{11}{12}$

P(A wins) $= P(A) + P(\bar{A})P(\bar{B})P(A) + ...$

$$= \frac{1}{12} + \frac{1}{12} \times \left(\frac{11}{12}\right)^2 + \frac{1}{12} \times \left(\frac{11}{12}\right)^4 + ...$$

$$= \frac{1}{12}\left[1 + \left(\frac{11}{12}\right)^2 + \left(\frac{11}{12}\right)^4 + ...\right] = \frac{1}{12}\left[\frac{1}{1 - \frac{11^2}{12^2}}\right]$$

$P(\text{A wins}) = \frac{1}{12} \times \left[\frac{12^2}{144 - 121}\right] = \frac{12}{23}$

P(B wins) = 1 – P (win A) $= 1 - \frac{12}{23} = \frac{11}{23}$