MATHEMATICS (SOLUTIONS)-2

1. Given; $R = \{(x, y) : x + 2y = 8\} \ \forall \ x, y \in N$

$\Rightarrow \quad R = \{(6, 1), (4, 2), (2, 3)\}$

Range of R is $\{1, 2, 3\}$

OR

Given that $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ and $A = \{1, 2, 3\}$

$\because$ $(1, 1), (2, 2), (3, 3) \in R$; Hence, R is reflexive.

Now; $(1, 2) \in R$ but $(2, 1) \in R$; Hence, R is not symmetric

and $(1, 2) \in R, (2, 3) \in R \Rightarrow (1, 3) \in R$

Hence, R is transitive

2. $f : R \to R$, given by $f(x) = [x]$

It is seen that $f(1.2) = [1.2] = 1$, $f(1.9) = [1.9] = 1$

$\therefore \quad f(1.2) = f(1.9)$, but $1.2 \quad 1.9$

$\Rightarrow$ f is not one-one

Now consider, $0.7 \in R$

It is known that $f(x) = [x]$ is always an integer. Thus, there does not exists any element $x \in R$ such that $f(x) = 0.7$.

$\therefore$ f is not onto.

3. Let $y = \dfrac{1}{2-\cos x} \ \forall \ x \in R$

$\Rightarrow \quad 2y - y\cos x = 1$

$\Rightarrow \quad \cos x = 2 - \dfrac{1}{y}$

Now, we know that $-1 \le \cos x \le 1$

$$\Rightarrow \quad -1 \le 2 - \frac{1}{y} \le 1 \quad \text{or} \quad -3 \le -\frac{1}{y} \le -1$$

$$\Rightarrow \quad 1 \le \frac{1}{y} \le 3 \quad \Rightarrow \quad \frac{1}{3} \le y \le 1 \quad \text{So, range is } \left[\frac{1}{3}, 1\right]$$

OR

Let $ARB \Rightarrow A \subset B$

Then, it is not necessary that B is a subset of A

i.e., $B \not\subset A$

$\Rightarrow$ B is not related to A

$\therefore$ R is not symmetric and hence R is not an equivalence relation

e.g., Let $X = \{1, 2, 3\}$

$P(X) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$

Clearly, $\{2\} \subset \{1, 2\}$ but $\{1, 2\} \not\subset \{2\}$

$\therefore$ R is not symmetric

4. Given $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$

$\Rightarrow$ $x - y = -1$

and $2x - y = 0$

Solving, we get $x = 1$ and $y = 2$

$x + y = 1 + 2 = 3$

5. Given, $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$

$\Rightarrow$ $12x + 14 = 32 - 42$

$\Rightarrow$ $x = -2$

OR

$7A - (I + A)^3 = 7A - (I^3 + A^3 + 3A^2I + 3I^2A)$

$= 7A - (I + A.A^2 + 3A + 3A)$

$= 7A - (I + A.A + 6A)$

$= 7A - (I + A^2 + 6A)$

$= 7A - (I + A + 6A)$

$= -I$

6. Given, $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{bmatrix}$

Cofactors of elements of 3rd row are

$$C_{31} = \begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix} = 18 - 10 = 8$$

$$C_{32} = \begin{vmatrix} 1 & -2 \\ 4 & 6 \end{vmatrix} = -(6 + 8) = -14$$

$$C_{33} = \begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} = -5 - 12 = -17$$

$\Rightarrow$ $|A| = a_{31}.C_{31} + a_{32}.C_{32} + a_{33}.C_{33}$

$= 3(8) + 5(-14) + 2(-17) = 24 - 70 - 34 = -80$

7. Let $I = \int_{-1}^{1} |x\cos\pi x|\,dx$

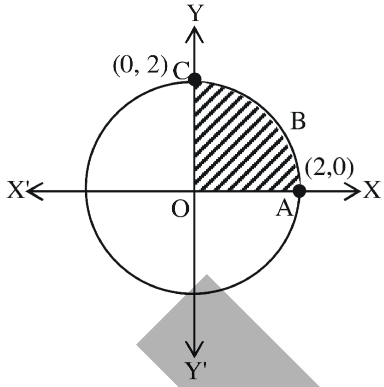
$$I = 2\int_{0}^{1} |x\cos\pi x|\,dx = 2\int_{0}^{1/2} (x\cos\pi x)dx - 2\int_{1/2}^{1} (x\cos\pi x)\,dx$$

$$= 2\left[\frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2}\right]_0^{1/2} - 2\left[\frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2}\right]_{1/2}^{1}$$

$$= 2\left[\frac{1}{2\pi}-\frac{1}{\pi^2}\right]-2\left[\frac{-1}{\pi^2}+\frac{-1}{2\pi}\right] = \frac{2}{\pi}$$

8. Given, $x^2 + y^2 = 4 \Rightarrow y = \sqrt{4-x^2}$

Shaded area = Area OABCO



$$= \int_0^2 y\,dx$$

$$= \int_0^2 \sqrt{4-x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= \left(2\cdot\frac{\pi}{2}\right)-0 = \pi \text{ sq. units}$$

9. Degree = 2 and Order = 2

$\therefore$ Sum = 4

OR

Given; y = A cos x - B sin x

$$\Rightarrow \quad \frac{dy}{dx} = -A\sin x - B\cos x$$

$$\therefore \quad \frac{d^2y}{dx^2} = -A\cos x + B\sin x$$

$$\text{or} \quad \frac{d^2y}{dx^2} = -y \Rightarrow \frac{d^2y}{dx^2}+y=0 \quad \frac{d^2y}{dx^2}+y=0$$

10. Let $\vec{a} = 2\hat{i}-3\hat{j}+6\hat{k}$

$$\Rightarrow \quad \text{Unit vector along given vector} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i}-3\hat{j}+6\hat{k}}{7}$$

$$\therefore \quad \text{Vector of magnitude 21 units is} = 21\left(\frac{2\hat{i}-3\hat{j}+6\hat{k}}{7}\right) = 6\hat{i}-9\hat{j}+18\hat{k}$$

11. Let $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b} = \hat{i}-2p\hat{j}+3\hat{k}$

Since $\vec{a}\,||\,\vec{b}$ $\quad \therefore \frac{3}{1} = \frac{2}{-2p} = \frac{9}{3}$

$$\Rightarrow p = \frac{-1}{3}$$

12. We know that $\vec{a} = |\vec{a}|\,(\ell\hat{i} + m\hat{j} + n\hat{k})$

Given ; $\ell = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

$m = \cos\dfrac{\pi}{2} = 0$

and $n = \cos\theta$

Now, $\ell^2 + m^2 + n^2 = 1$

$\dfrac{1}{2} + 0 + \cos^2\theta = 1$

$\Rightarrow$ (θ is acute)

$\therefore$ $\vec{a} = 5\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\hat{i} + 0\,\hat{j} + \dfrac{1}{\sqrt{2}}\hat{k}\right)$

$\Rightarrow$ $\vec{a} = 5(\hat{i} + \hat{k})$

13. Given $\dfrac{3-x}{5} = \dfrac{y+4}{4} = \dfrac{2z-6}{4}$

$\Rightarrow$ $\dfrac{x-3}{-5} = \dfrac{y+4}{4} = \dfrac{z-3}{2}$

Vector equation is $\vec{r} = (3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-5\hat{i} + 4\hat{j} + 2\hat{k})$

14. Given, $2x - 2y - z = -3$

and $4x - 4y - 2z + 5 = 0$ $\Rightarrow$ $2x - 2y - z = \dfrac{-5}{2}$

Distance between the parallel planes is given as:

$$d = \left|\frac{d_2 - d_1}{\sqrt{A^2 + B^2 + C^2}}\right| = \left|\frac{-\frac{5}{2} + 3}{\sqrt{4+4+1}}\right| = \frac{1}{6}\text{ units}$$

15. Equation of plane in intercept form is given as:

$\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

$\therefore$ $\dfrac{x}{-4} + \dfrac{y}{2} + \dfrac{z}{3} = 1$

$\Rightarrow$ $-3x + 6y + 4z = 1$

16. Given, P(A hits) = 0.4, P(B hits) = 0.3, P(C hits) = 0.2

$\therefore$ P (A does not hit) = 1 – 0.4

= 0.6

Also, P(B does not hit) = 0.7 and P(C does not hit) = 0.8

$\Rightarrow$ P (2 hits) = $P(\bar{A} \cap B \cap C) + P(A \cap \bar{B} \cap C) + P(A \cap B \cap \bar{C})$

$= 0.6 \times 0.3 \times 0.2 + 0.4 \times 0.7 \times 0.2 + 0.4 \times 0.3 \times 0.8$

$= 0.188$

17. x = awarded members for honesty

y = awarded members for helping (Co-operation)

z = awarded member for supervision

According to the given eqestion

$x + y + z = 12$ ……….(1)

$3(y + z) + 2x = 33$

$\Rightarrow$ $2x + 3y + 3z = 33$ ……….(2)

and $x + z = 2y$

$\Rightarrow$ $x - 2y + z = 0$ ……….(3)

All the above three equations can be written in matrix form as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$

$\Rightarrow$ A . X = B ……...(4)

Here, $|A| = 1(3 + 6) - 1(2 - 3) + 1(-4 - 3)$

$= 9 + 1 - 7 = 3 \neq 0$

Here, A is non-singular and so its inverse exists. Now,

$A_{11} = 9, A_{12} = 1, A_{13} = -7$

$A_{21} = -3, A_{22} = 0, A_{23} = 3$

$A_{31} = 0, A_{32} = -1, A_{33} = 1$

$A_{11} = 9, A_{21} = -3, A_{31} = 0$ …..…(5)

Now, $\text{adj } A = \left[A_{ij}\right]^T = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$

$\therefore$ $A^{-1} = \dfrac{\text{adj}A}{|A|} = \dfrac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$ …..…(6)

So, $X = A^{-1}. B = \dfrac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix} . \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 108 - 99 + 0 \\ 12 + 0 + 0 \\ -84 + 99 + 0 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

or $x = 3, y = 4, z = 5$ ……..(7)

(i) **(a)** $x + y + z = 12$ (from (1))

$2x + 3y + 3z = 33$ (from (2))

$x - 2y + z = 0$ (from (3))

(ii) **(c)** $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 33 \\ 12 \\ 0 \end{bmatrix}$ (from (4))

(iii) **(b)** $A_{11} = 9, A_{21} = -3, A_{31} = 0$ (from (5))

(iv) **(a)** $A^{-1} = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$ (from (6))

(v) **(d)** $x = 3, y = 4, z = 5$ (from (7))

18. Let us define the events as:

E_1 = ghee purchased from shop X,

E_2 = ghee purchased from shop Y,

and A = getting adulterated ghee

$\Rightarrow$ $P(E_1) = P(E_2) = \frac{1}{2}$

and $P(A/E_1) = \frac{40}{70} = \frac{4}{7}$

$P(A/E_2) = \frac{60}{110} = \frac{6}{11}$ …….(1)

Now; $P(E_1).(A/E_1) = \frac{1}{2} \times \frac{4}{7} = \frac{2}{7}$ …….(2)

and $P(E_2).P(A/E_2) = \frac{1}{2} \times \frac{6}{11} = \frac{3}{11}$

$\Rightarrow$ $P(E_1).\ P(A/E_1) + P(E_2).\ P(A/E_2) = \frac{2}{7} + \frac{3}{11} = \frac{22+21}{77} = \frac{43}{77}$ …….(3)

Now; $P(E_2/A) = \frac{P(E_2).P(A/E_2)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2)}$

$= \frac{3/11}{43/77} = \frac{3}{11} \times \frac{77}{43} = \frac{21}{43}$ …….(4)

and $P(E_1/A)$

$= \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2)} = \frac{2/7}{43/77} = \frac{2}{7} \times \frac{77}{43} = \frac{22}{43}$ …….(5)

From equations (4) and (5), we have;

$$P(E_1/A) + P(E_2/A)$$

$$= \frac{22}{43} + \frac{21}{43} = \frac{43}{43} = 1 \qquad \ldots\ldots(6)$$

(i) **(a)** Required Probability

$$= P(A/E_2) = \frac{6}{11} \qquad \text{(from (1))}$$

(ii) **(b)** Required Probability

$$= P(E_1).P(A/E_1) = \frac{2}{7} \qquad \text{(from (2))}$$

(iii) **(d)** Required Probability

$$= P(E_1).P(A/E_1) + P(E_2).\ P(A/E_2) = \frac{43}{77} \qquad \text{(from (3))}$$

(iv) **(b)** Required Probability

$$= P(E_2/A) = \frac{21}{43} \qquad \text{(from (4))}$$

(v) **(c)** We have;

$$\sum_{i=1}^{2} P\left(E_i / A\right) = P(E_1/A) + P(E_2/A) = 1 \qquad \text{(from (6))}$$

19. Let $\sin^{-1}\left(\sin\frac{5\pi}{3}\right) = \theta$

$\Rightarrow \quad \sin\theta = \sin\left(\frac{5\pi}{3}\right)$

or $\quad \sin\theta = \sin\left(2\pi - \frac{\pi}{3}\right)$

$\Rightarrow \quad \sin\theta = -\sin\frac{\pi}{3} \qquad [\because \sin(2\pi - \theta) = -\sin\theta]$

or $\quad \sin\theta = \sin\left(-\frac{\pi}{3}\right)$

$\Rightarrow \quad \theta = -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

20. Given, $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = 0$

$\Rightarrow \quad (1 + a)\,[(1 + a)^2 - 1] - 1\,[1 + a - 1] + 1[1 - (1 + a)] = 0$

$\Rightarrow \quad (1 + a)\,[a^2 + 2a] - 1\,[a] + 1\,[-a] = 0$

$\Rightarrow \quad 3a^2 + a^3 = 0$

Solving, we get $a = -3$ or $a = 0$

OR

Given, $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$\Rightarrow \quad A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\therefore \quad A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

21. At $x = 0$, $f(0) = a$ …..(1)

$\text{LHL} = \lim\limits_{x \to a} f(x) = \lim\limits_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \frac{1 - \cos 4(-h)}{(-h)^2} = \lim_{h \to 0} \frac{1 - \cos 4h}{h^2} = \lim_{h \to 0} \frac{2\sin^2 2h}{h^2} \times \frac{4}{4} = \lim_{h \to 0} 8\left(\frac{\sin 2h}{2}\right)^2 = 8 \quad \text{…..(2)}$$

Since f(x) is continuous function, from (1) and (2); $a = 8$

22. Given, $f(x) = x^2 + 4x + 5$

$\Rightarrow \quad f'(x) = 2x + 4$

For maxima or minima, $f'(x) = 0$

$\Rightarrow \quad 2x + 4 = 0$

or $\quad x = -2$

Also $f''(x) = 2 > 0 \quad \therefore$ f(x) is minimum at $x = -2$

$\Rightarrow \quad f(-2) = (-2)^2 + 4(-2) + 5 = 1$

23. Given ; $f(x) = \int\limits_0^x t \sin t \, dt = \int\limits_0^x \cos t \, dt$

$\Rightarrow \quad f(x) = \left[-t\cos t + \sin t\right]_0^x$

or $\quad f(x) = -x\cos x + \sin x$

$\Rightarrow \quad f(x) = -(x.(-\sin x) + \cos x.1) + \cos x = x \sin x$

OR

$$I = \int\limits_e^{e^2} \frac{dx}{x \log x}$$

Put $\log x = t$ — $x = e^2, \quad t = 2$

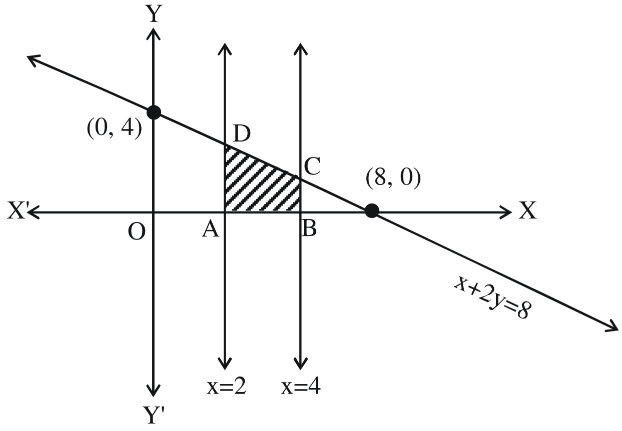
$\Rightarrow \frac{1}{x}dx = dt$ — $x = e, \quad t = 1$

$$I = \int\limits_1^2 \frac{1}{t}\, dt = \left[\log t\right]_1^2 = \log 2$$

24. Given, $2y = -x + 8$

$\Rightarrow \quad y = \dfrac{-x}{2} + 4$

Now, Shaded area = Area ABCDA



$$= \int_2^4 y\, dx$$

$$= \int_2^4 \left(\frac{-x}{2} + 4\right) dx$$

$$= \left[\frac{-x^2}{4} + 4x\right]_2^4 = (-4 + 16) - (-1 + 8) = 12 - 7 = 5 \text{ sq. units}$$

25. Given, $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

$$\Rightarrow \frac{dy}{dx} + \sec^2 x . y = \sec^2 x \tan x$$

This is of the form $\dfrac{dy}{dx} + Py = Q$

where $P = \sec^2 x$, $Q = \sec^2 x \tan x$

$\therefore \quad$ I.F. $= e^{\int P dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$

Its solution is given by

$$y.\ IF = \int Q.\ IF\ dx + C$$

$$\Rightarrow \quad y.e^{\tan x} = \int \sec^2 x \tan x \cdot e^{\tan x}\ dx + C$$

$$\therefore \quad y.e^{\tan x} = \tan x\ .\ e^{\tan x} - e^{\tan x} + C$$

26. Given $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$

$$\Rightarrow \quad 2\vec{a} - \vec{b} + 3\vec{c} = 2(\hat{i} + \hat{j} + \hat{k}) - (4\hat{i} - 2\hat{j} + 3\hat{k}) + 3(\hat{i} - 2\hat{j} + \hat{k})$$

$$= \hat{i} - 2\hat{j} + 2\hat{k} = \vec{d} \text{ (let)}$$

$\therefore$ Vector parallel to $\vec{d}$ and having magnitude 6 units

$$= 6\left(\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1 + 4 + 4}}\right)$$

$$= 2(\hat{i} - 2\hat{j} + 2\hat{k})$$

27. Line through the point A(4, 3, 2) and B(1, –1, 0) is :

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\Rightarrow \quad \frac{x - 4}{-3} = \frac{y - 3}{-4} = \frac{z - 2}{-2}$$

Similarly, line passing through C(1, 2, –1) ad D(2, 1, 1) is :

$$\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z+1}{2}$$

Direction ratios of line perpendicular to AB and CD is :

$$\frac{a}{(-4)(2)-(-1)(-2)}=\frac{b}{(1)(-2)-(-3)(2)}=\frac{c}{(-3)(-1)-(1)(-4)}$$

$\Rightarrow$ $\frac{a}{-10}=\frac{b}{4}=\frac{c}{7}$

$\therefore$ Equation of line passing through (1, –1, 1) and having direction ratios –10, 4, 7 is :

$$\frac{x-1}{-10}=\frac{y+1}{4}=\frac{z-1}{7}$$

28. The sample space of the experiment is :

S = {(T, H), (T, T), (H, 1), (H, 2), (H, 3),(H, 4), (H, 5), (H, 6)}

Let A be the event that die shows number greater than 3 and B be the event that there is atleast 1 head

$\therefore$ A = {(H, 4), (H, 5), (H, 6)}

and B = {(T, H), (H, 1), (H, 2), (H, 3),(H, 4), (H, 5), (H, 6)}

$\Rightarrow$ $A \cap B$ = {(H, 4), (H, 5), (H, 6)}

$P(A \cap B) = \frac{1}{12}\times 3=\frac{1}{4}$

and $P(B) = \frac{1}{4}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}$

$=\frac{3}{4}$

$\Rightarrow$ $P(A/B) = \frac{P(A\cap B)}{P(B)}=\frac{1/4}{3/4}=\frac{1}{3}$

29. Consider (a, b) R (a, b) where $(a, b) \in A \times A$

$\therefore$ a + b = b + a

Hence, R is reflexive relation

Now consider (a, b) R (c, d) given by (a, b), $(c, d) \in A \times A$. Then,

a + d = b + c $\Rightarrow$ c + b = d + a

$\Rightarrow$ (c, d) R (a, b)

$\therefore$ R is symmetric relation

Let (a, b) R (c, d) and (c, d) R (e, f)

Where (a, b) (c, d), $(e, f) \in A \times A$

$\Rightarrow$ a + d = b + c …. (1) $\Rightarrow$ (1) + (2) gives:

and c + f = d + e ….. (2)

a + f = b + e

$\Rightarrow$ (a, b) R (e, f)

R is transitive relation

Hence; R is an equivalence relation

Now,[(2, 5)] = {(1, 4), (2,5), (3, 6), (4, 7), (5, 8), (6, 9)} [$\because$ Let (x,y) R (2,5) $\Rightarrow$ x+5 = y +2 or x+3 = y]

30. Let $y = \sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right)$

$$= \sin^{-1}\left(\frac{2.6^x}{1+(6^x)^2}\right)$$

Put $6^x = \tan\theta$, $\theta = \tan^{-1} 6^x$

$\Rightarrow \quad y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$\therefore \quad y = \sin^{-1}(\sin 2\theta) \quad$ or $\quad y = 2\theta$

$\Rightarrow \quad y = 2\tan^{-1} 6^x$

or $\quad \frac{dy}{dx} = \frac{2}{1+36^x} \times 6^x \log_e 6$

31. $x = a\cos^3\theta$; $y = a\sin^3\theta$

$\Rightarrow \quad \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) \quad$ ....(1)

and $\quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta \quad$ ....(2)

(2) $\div$ (1) gives:

$$\frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{d\theta}[-\tan\theta].\frac{d\theta}{dx} = -\sec^2\theta.\frac{d\theta}{dx}$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{-\sec^2\theta}{-3a\cos^2\theta.\sin\theta} \quad$ [from (1)]

or $\quad \frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\theta\sin\theta}$

At $\theta = \frac{\pi}{6}$; $\frac{d^2y}{dx^2} = \frac{1}{3a\times\left(\frac{\sqrt{3}}{2}\right)^4\times\frac{1}{2}} = \frac{32}{27a}$

OR

$(ax+b)e^{y/x} = x$

$\Rightarrow \quad e^{y/x} = \frac{x}{ax+b} \quad$ .....(1)

Taking log both sides;

$$\frac{y}{x} = \log x - \log(ax+b)$$

Differentiating w.r.t. x; we have:

$$\frac{x\frac{dy}{dx}-y}{x^2}=\frac{1}{x}-\frac{a}{ax+b}$$

$\Rightarrow$ $$x\frac{dy}{dx}-y=\frac{bx}{ax+b}$$

From (1);

$$x\frac{dy}{dx}-y=b\cdot e^{y/x} \qquad \text{.....(2)}$$

Differentiating w.r.t. x again;

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-\frac{dy}{dx}=\frac{be^{y/x}\left(x\frac{dy}{dx}-y\right)}{x^2}$$

From eq. (2), we have:

$$x^3\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^2$$

32. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Differentiating w.r.t x;

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0 \qquad \text{or} \qquad \frac{dy}{dx}=\frac{b^2x}{a^2y} \qquad \text{....(1)}$$

$\Rightarrow$ Slope of tangent at $(\sqrt{2}a, b)=\frac{\sqrt{2}\,b}{a}$ (from (1))

$\therefore$ Slope of normal at $(\sqrt{2}a, b)=\frac{-a}{\sqrt{2}\,b}$

$\Rightarrow$ Equation of tangent is :

$$y-b=\frac{\sqrt{2}b}{a}(x-\sqrt{2}a)$$

or $\sqrt{2}bx-ay=ab$

and Equation of normal is :

$$y-b=\frac{-a}{\sqrt{2}b}(x-\sqrt{2}a)$$

$$ax+\sqrt{2}by=\sqrt{2}(a^2+b^2)$$

33. Let $I = \int_0^{\pi/4} \left(\frac{\sin x + \cos x}{3 + \sin 2x} \right) dx$

$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{4 - (\sin x - \cos x)^2} dx$

Put $\sin x - \cos x = t$; $x = 0, \quad t = -1$

$(\cos x + \sin x)dx = dt$; $x = \frac{\pi}{4}, \quad t = 0$

$\therefore \quad I = \int_{-1}^{0} \frac{1}{4 - t^2} dt$

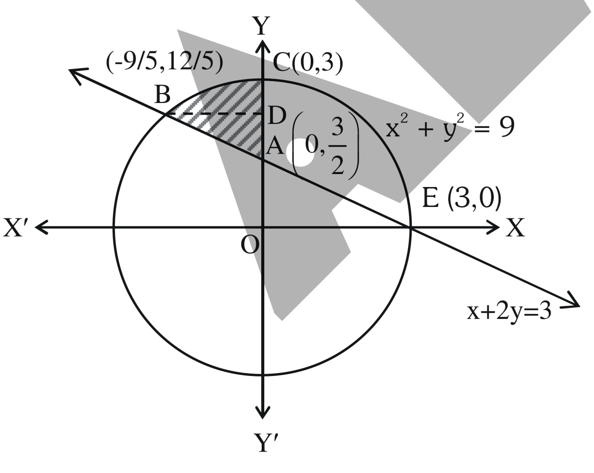
or $\quad I = \left[\frac{1}{4} \log \frac{2+t}{2-t} \right]_{-1}^{0}$

$\Rightarrow \quad I = \frac{1}{4}\left[\log 1 - \log \frac{1}{3} \right]$

or $\quad I = \frac{1}{4} \log 3$

34. Given ; $x^2 + y^2 = 9$
and $x + 2y = 3$

Their point of intersection is $\left(-\frac{9}{5}, \frac{12}{5} \right)$



Required shaded area = Area (ABDA) + Area (BDCB)

$$= \int_{3/2}^{12/5} x\,dy + \int_{12/5}^{3} x\,dy$$

$$= \int_{3/2}^{12/5} (3-2y)\,dy + \int_{12/5}^{3} \sqrt{9-y^2}\,dy$$

$$= \left[3y - y^2\right]_{3/2}^{12/5} + \left[\frac{y}{2}\sqrt{9-y^2} + \frac{9}{2}\sin^{-1}\frac{y}{3}\right]_{12/5}^{3}$$

$$= \left[\left(\frac{36}{5} - \frac{144}{25}\right) - \left(\frac{9}{2} - \frac{9}{4}\right)\right] + \left[\frac{9}{2}.\frac{\pi}{2} - \left(\frac{6}{5}.\frac{9}{5} + \frac{9}{2}\sin^{-1}\frac{4}{5}\right)\right]$$

$$= \frac{-81}{100} + \frac{9\pi}{4} - \frac{54}{25} - \frac{9}{2}\sin^{-1}\frac{4}{5}$$

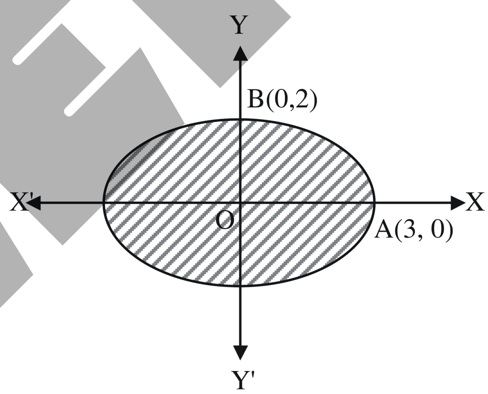
$$= \left(\frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\frac{4}{5} - \frac{297}{100}\right) \text{ sq. units}$$

OR

Given $4x^2 + 9y^2 = 36$

$\Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{4} = 1$

Area bounded by ellipse = 4 × Area (OABO)



$$= 4 \times \int_0^3 y\,dx$$

$$= 4 \times \int_0^3 \frac{2}{3}\sqrt{9-x^2}\,dx$$

$$= \frac{8}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3$$

$$= \frac{8}{3}\left[\frac{9}{2}.\frac{\pi}{2}\right]$$

$= 6\pi$ sq. units

35. The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)} \qquad \text{.......(1)}$$

Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

From(1);

$$v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v}$$

$$\therefore\ x\frac{dv}{dx} = \frac{1}{\cos v}$$

or $\quad \cos v\, dv = \frac{dx}{x}$

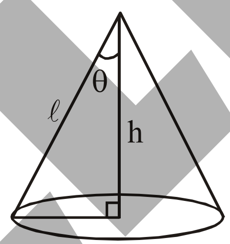
Integrating both sides; we have :

$$\int \cos v\ dv = \int \frac{1}{x}\,dx$$

$\Rightarrow \quad \sin v = \log|x| + C$ or $\sin\frac{y}{x} = \log|x| + C$

36. Let radius of cone = r
height of cone = h
slant height of cone = ℓ
and semi-vertical angle = α

Volume of cone V = $\frac{1}{3}\pi r^2 h = \frac{\pi\ell^3}{3}\sin^2\theta\cos\theta$ [$\because$ h = ℓ cos θ , r = ℓ sin θ]



$$\frac{dV}{d\theta} = \frac{\pi\ell^3}{3}\left[2\sin\theta\cos^2\theta - \sin^3\theta\right]$$

for maxima and minima $\frac{dV}{d\theta} = 0$

$$\frac{dV}{d\theta} = \frac{\pi\ell^3}{3}\left[2\sin\theta\cos^2\theta - \sin^3\theta\right] = 0$$

$\Rightarrow \quad \sin\theta\left[2\cos^2\theta - (1-\cos^2\theta)\right] = 0$

$\Rightarrow \quad 3\cos^2 - 1 = 0 \qquad [\because \sin\theta \neq 0]$

$$\cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{3}}$$

$$\frac{d^2V}{d\theta^2} = \frac{\pi\ell^3}{3}\left[2\cos^3\theta - 7\sin^2\theta\cos\theta\right]$$

$\frac{d^2V}{d\theta^2}$ is negative at $\theta = \cos^{-1}\frac{1}{\sqrt{3}}$

Hence V is maximum at $\theta = \cos^{-1}\frac{1}{\sqrt{3}}$

OR

Let ABC be an isosceles triangle inscribed in the circle with radius a such that

AB = AC.

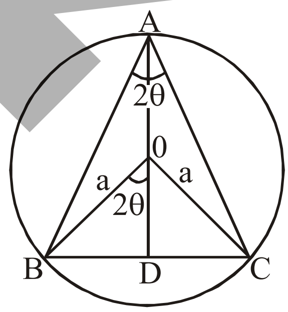
AD = AO + OD = a + a$\cos^2\theta$ and BC = 2BD = 2a$\sin^2\theta$ (see figure)

Therefore, area of the triangle ABC i.e. $\Delta = \frac{1}{2}BC.AD$

$$= \frac{1}{2} 2a\sin 2\theta.(a + a\cos 2\theta)$$

$$= a^2\sin^2\theta\ (1 + \cos^2\theta)$$

$$\Rightarrow \qquad \Delta = a^2 \sin 2\theta + \frac{1}{2}a^2 \sin 4\theta$$



Therefore, $\frac{d\Delta}{d\theta} = 2a^2\cos 2\theta + 2a^2 \cos 4\theta$

$$= 2a^2(\cos 2\theta + \cos 4\theta)$$

$\frac{d\Delta}{d\theta} = 0 \Rightarrow \cos 2\theta = -\cos 4\theta = \cos(\pi - 4\theta)$

Therefore, $2\theta = \pi - 4\theta \Rightarrow \theta = \frac{\pi}{6}$

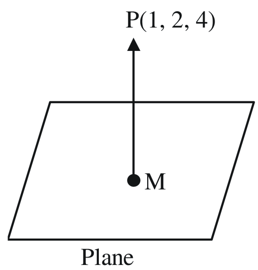
$$\frac{d^2\Delta}{d\theta^2} = 2a^2(-2\sin 2\theta - 4\sin 4\theta) < 0 \left(\text{at } \theta = \frac{\pi}{6}\right)$$

Therefore, Area of triangle is maximum when $\theta = \frac{\pi}{6}$.

37. Given equation of plane is :

$2x + y - 2z + 3 = 0$ ……(1)

Direction ratios of normal to the plane are 2, 1, –2



$\therefore$ Equation of line PM is :

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-4}{-2} = \lambda(\text{say})$$

$\therefore$ Co-ordinates of M = $(2\lambda + 1, \lambda + 2, -2\lambda + 4)$

Since M lies on plane (1); we have :

$\therefore$ $2(2\lambda + 1) + (\lambda + 2) - 2(-2\lambda + 4) + 3 = 0$

$\Rightarrow \quad \lambda = \frac{1}{9}$

$\therefore \quad$ Foot of perpendicular $= \left(\frac{2}{9}+1, \frac{1}{9}+2, \frac{-2}{9}+4\right) = \left(\frac{11}{9}, \frac{19}{9}, \frac{34}{9}\right)$

$\Rightarrow \quad$ Length of perpendicular from (1, 2, 4) i.e.

$$PM = \left|\frac{2(1)+2-2(4)+3}{\sqrt{4+1+4}}\right| = \frac{1}{3}\text{ units}$$

OR

Any point on the line $\vec{r} = 2\hat{i}-4\hat{j}+2\hat{k}+\lambda(3\hat{i}+4\hat{j}+2\hat{k})$ is $x = 2 + 3\lambda$, $y = -4 + 4\lambda$, $z = 2 + 2\lambda$

Equation of plane is

$$\vec{r}.(\hat{i}-2\hat{j}+\hat{k}) = 0$$

$\Rightarrow \quad x - 2y + z = 0$

Point lies on plane

$\Rightarrow \quad (2 + 3\lambda) - 2(-4 + 4\lambda) + (2 + 2\lambda) = 0$

$\Rightarrow \quad \lambda = 4$

$\Rightarrow \quad$ Point of intersection is (14, 12, 10)

Distance of point (2, 12, 5) from (14, 12, 10) is

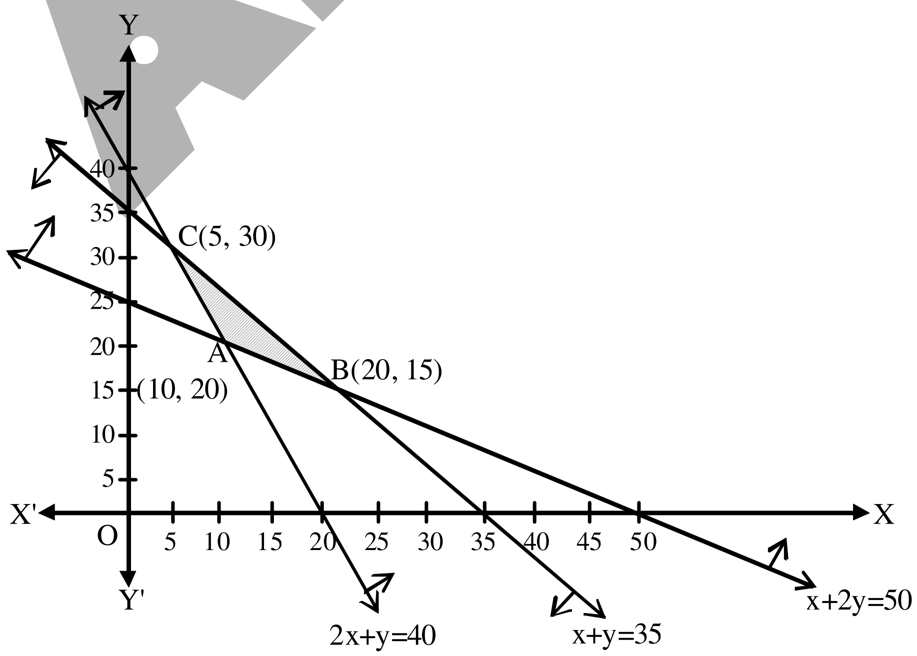
$$= \sqrt{(14-2)^2+(12-12)^2+(10-5)^2} = 13 \text{ units}$$

38. Minimize $\quad Z = 5x + 4y$

subject to constraints :

$$x + 2y \geq 50 \;;\; 2x + y \geq 40 \;;\; x + y \leq 35 \;;\; x, y \geq 0$$

Plot the straight lines on the graph as shown :



Corner points of bounded feasible region are A(10, 20), B(20, 15), C(5, 30)

Corner points	$Z = 5x + 4y$
A(10, 20)	130
B(20, 15)	160
C(5, 30)	145

Minimum value of Z is 130.

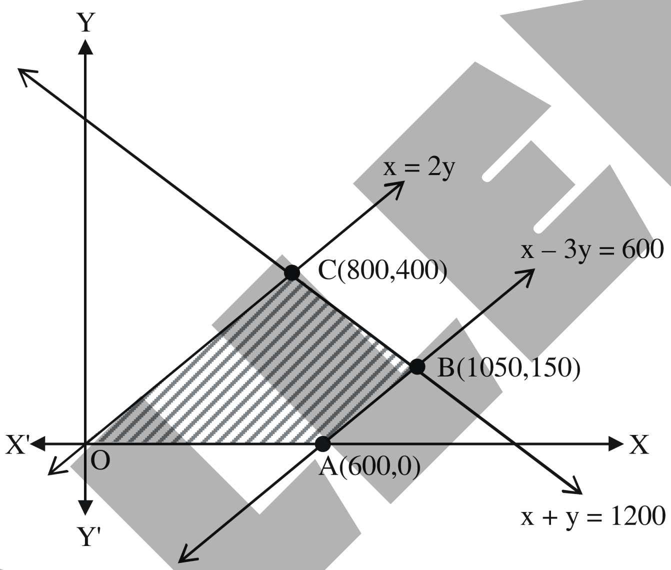
OR

Given objective function : $Z = 12x + 16y$

subject to constraints :

$x + y \le 1200$; $x \ge 2y$; $x - 3y \le 600$; $x, y \ge 0$

Plot the straight lines in graph as shown



Corner points of bounded feasible region are A(600, 0), B(1050, 150), C(800, 400), O(0, 0)

Corner points	$Z = 12x + 16y$
A(600,10)	7360
B(1050, 150)	15000
C(800, 400)	16000
O(0, 0)	0

Maximum value of Z is 16000