MATHEMATICS (SOLUTIONS) -1

1. Here;

$(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$

$\therefore$ R is not transitive

OR

$f = \{(1, 4), (2, 5), (3, 6)\}$

$\therefore f(1) = 4, f(2) = 5, f(3) = 6$

Since different elements have different images; so f is one-one function.

2. Two equivalence relations are there in all;

$\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ and $\{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3,2)\}$

3. Required number of mappings = 0

[Since both the sets have different number of elements]

OR

Given relation, $R = \{(a, a), (b, c), (a, b)\}$

So, required minimum number of ordered pairs to be added are (b, b), (c, c) and (a, c).

4. Order of matrix '7X' is '$2 \times n$' and order of matrix '5Z' is '$2 \times p$' i.e. '$2 \times n$' [$\because n = p$]

Thus, matrix $(7X - 5Z)$ has order '$2 \times n$'.

5. We know that;

$|\text{adj } A| = |A|^{n-1}$; where 'n' represents the order of matrix.

Hence; $|\text{adj } A| = |A|^{3-1} = |A|^2$

OR

$$\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} + \sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

6. $a_{32}.A_{32}$

$$= -5\begin{vmatrix}2 & 5\\ 6 & 4\end{vmatrix}$$

$= -5\,(8 - 30)$

$= 110$

7. $$\int_{-\pi/4}^{\pi/4}\frac{dx}{1+\cos 2x} = \int_{-\pi/4}^{\pi./4}\frac{dx}{2\cos^2 x}$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4}\sec^2 x\ dx = \frac{1}{2}\times 2\int_{0}^{\pi/4}\sec^2 x\ dx$$

$$= \left[\tan x\right]_0^{\pi/4} = 1$$

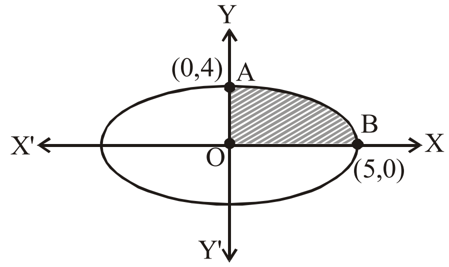
OR

$$\int \frac{x+3}{(x+4)^2} e^x dx = \int e^x \left[\frac{(x+4)-1}{(x+4)^2}\right] dx$$

$$= \int e^x \left[\frac{1}{(x+4)} + \left\{\frac{-1}{(x+4)^2}\right\}\right]$$

$$= e^x \left[\frac{1}{x+4}\right] + C$$

8.



Required Area $= 4 \times \int_0^5 \frac{4}{5}\sqrt{5^2 - x^2}\, dx$

$$= \frac{16}{5}\left[\frac{x}{2}\sqrt{5^2 - x^2} + \frac{5^2}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_0^5$$

$$= \frac{16}{5}\left[0 + \frac{5^2}{2} \times \frac{\pi}{2} - 0 - 0\right]$$

$= 20\,\pi$ sq. units

9. $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$

$\Rightarrow \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$ {on squaring}

Hence; degree = 2

OR

Number of arbitrary constants = 0

10. Let $\vec{r} = \vec{a} + \vec{b} = (2\hat{i} - \hat{j} + 2\hat{k}) + (-\hat{i} + \hat{j} + 3\hat{k}) = \hat{i} + 5\hat{k}$

$$\Rightarrow \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\hat{i} + 5\hat{k}}{\sqrt{26}} = \frac{1}{\sqrt{26}}\hat{i} + \frac{5}{\sqrt{26}}\hat{j}$$

11. Area of Δ OAB

$$= \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$$

$$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|-9\hat{i} + 2\hat{j} + 12\hat{k}\right|$$

$$= \frac{1}{2}\sqrt{229} \text{ sq. units}$$

12. $(\vec{a} + \vec{b} + \vec{c}).(\vec{a} + \vec{b} + \vec{c}) = \vec{0} \cdot \vec{0}$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

$\Rightarrow 3 + 2\ (\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

or $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = -3/2$

13. Direction cosines of $(2\hat{i} + 2\hat{j} - \hat{k})$ are :

$$\frac{2}{\sqrt{4+4+1}}, \frac{2}{\sqrt{4+4+1}}, \frac{-1}{\sqrt{4+4+1}}$$

$$= \frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$$

14. The required vector equation of line is given as :

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda\ (\hat{i} + 2\hat{j} - 5\hat{k})$

15. P [Red transferred and red drawn or black transferred and red drawn]

$$= \left(\frac{3}{8} \times \frac{7}{11}\right) + \left(\frac{5}{8} \times \frac{6}{11}\right)$$

$$= \frac{51}{88}$$

16. $P(A \cup B) = 0.6,\ P(A \cap B) = 0.3$

$\because\ P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow\ P(A \cup B) + P(A \cap B) = [1 - P(\overline{A})] + [1 - P(\overline{B})]$

or $P(\overline{A}) + P(\overline{B}) = 2 - (0.6 + 0.3)$

$= 1.1$

17. Let E be the event that the doctor visits the patient late and let T_1, T_2, T_3, T_4 be the events that the doctor comes by train, bus, scooter and other means of transport respectively.

Then; $P(T_1) = \frac{3}{10},\ P(T_2) = \frac{1}{5},\ P(T_3) = \frac{1}{10},\ P(T_4) = \frac{2}{5}$ (given)

and $P(E/T_1) = \frac{1}{4},\ P\left(E/T_2\right) = \frac{1}{3},\ P\left(E/T_3\right) = \frac{1}{12},\ P\left(E/T_4\right) = 0$ (given)

Now; $P(T_1).P(E/T_1) + P(T_2).P(E/T_2) + P(T_3).P(E/T_3) + P(T_4).P(E/T_4)$

$$= \left(\frac{3}{10}\times\frac{1}{4}\right) + \left(\frac{1}{5}\times\frac{1}{3}\right) + \left(\frac{1}{10}\times\frac{1}{12}\right) + \left(\frac{2}{5}\times 0\right)$$

$= 18/120$ ...(1)

$$\Rightarrow \quad P(T_1/E) = \frac{P(T_1).P(E/T_1)}{P(T_1).P(E/T_1) + P(T_2).P(E/T_2) + P(T_3).P(E/T_3) + P(T_4).P(E/T_4)}$$

$$= \frac{\frac{3}{10}\times\frac{1}{4}}{\frac{18}{120}} = \frac{3}{40}\times\frac{120}{18} = \frac{1}{2} \quad \text{(from (1))} \qquad ...(2)$$

Similarly; $P(T_2/E) = \dfrac{P(T_2).P(E/T_2)}{18/120}$ (from (1))

$$= \frac{\frac{1}{5}\times\frac{1}{3}}{\frac{18}{120}} = \frac{1}{15}\times\frac{120}{18} = \frac{4}{9} \qquad ...(3)$$

$P(T_3/E) = \dfrac{P(T_3).P(E/T_3)}{18/120}$ (from (1))

$$= \frac{\frac{1}{10}\times\frac{1}{12}}{\frac{18}{120}} = \frac{1}{120}\times\frac{120}{18} = \frac{1}{18} \qquad ...(4)$$

and $P(T_4/E) = \dfrac{P(T_4).P(E/T_4)}{18/120}$ (from (1))

$= 0$ ...(5)

(i) (b) Required probability

$= P(E/T_3) = 1/12 = 0.083$

(ii) (c) Required probability

$= 1 - P(T_1/E) = 1 - 1/2 = 1/2 = 0.5$

(iii) (d) Required probability

$= P(T_2).P(E/T_2)$

$$= \frac{1}{5} \times \frac{1}{3} = \frac{1}{15} = 0.066$$

(iv) (c) $\sum_{i=1}^{4} P(T_i / E)$

$= P(T_1/E) + P(T_2/E) + P(T_3/E) + P(T_4/E)$

$$= \frac{1}{2} + \frac{4}{9} + \frac{1}{18} + 0 \quad \text{(from eqn's (2), (3), (4), (5))}$$

$$= \frac{9+8+1}{18} = \frac{18}{18} = 1$$

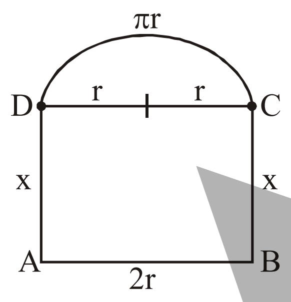
(v) (a) Required probability

$= P(T_1).P(E/T_1) + P(T_2).P(E/T_2) + P(T_3).P(E/T_3) + P(T_4).P(E/T_4)$

$$= \frac{18}{120} \quad \text{(from (1))}$$

$= 0.15$

18.



Let radius of semi-circle = r

$\Rightarrow$ One side of rectangle = 2r = length,

and other side of rectangle = x = breadth

Let P = Perimeter of the window

$\Rightarrow$ P = 10m (given)

Now; $2x + 2r + \frac{1}{2}(2\pi r) = 10$

$\Rightarrow$ $2x = 10 - r(\pi + 2)$ ...(1)

or $2x + r(\pi + 2) = 10$ ...(2)

Let A be the area of the figure; then :

A = Area of semi-circle + Area of rectangle

$= \frac{1}{2}\pi r^2 + 2rx$

$= \frac{1}{2}\pi r^2 + r\,[10 - r\,(\pi + 2)]$ (from (1))

$= \frac{\pi r^2}{2} + 10r - r^2\pi - 2r^2$

$\Rightarrow A = 10r - \frac{\pi r^2}{2} - 2r^2$ ...(3)

On differentiating twice w.r.t. 'r', we get :

$\frac{dA}{dr} = 10 - \pi r - 4r$

and $\frac{d^2A}{dr^2} = -\pi - 4$

For maxima or minima; $\frac{dA}{dr} = 0$

$\Rightarrow \quad 10 - \pi r - 4r = 0$

or $\quad r = \frac{10}{4+\pi}$ ...(4)

Now, $\frac{d^2A}{dr^2} = -(\pi + 4) < 0$

Hence; A has local maximum when $r = \frac{10}{4+\pi}$

$\therefore$ Radius of semi-circle $= \frac{10}{4+\pi}$

$\Rightarrow$ Length of rectangle $= 2r = \frac{20}{4+\pi}$ ...(5)

and Breadth of rectangle = x

$$= \frac{1}{2}\left[10 - r(\pi+2)\right] = \frac{1}{2}\left[10 - \frac{10}{(4+\pi)}(\pi+2)\right] = \frac{10\pi + 40 - 10\pi - 20}{2(\pi+4)}$$

$\Rightarrow \quad x = \frac{20}{2(\pi+4)} = \frac{10}{\pi+4}$ ...(6)

And the maximum area of the window is given as :

$$A = 10r - \frac{\pi r^2}{2} - 2r^2 \qquad \text{(from (3))}$$

$$= 10 \times \left[\frac{10}{4+\pi}\right] - \frac{\pi}{2} \times \left[\frac{10}{4+\pi}\right]^2 - 2 \times \left[\frac{10}{4+\pi}\right]^2$$

$$= \frac{100}{4+\pi} - \frac{50\pi}{(4+\pi)^2} - \frac{200}{(4+\pi)^2}$$

$$= \frac{100\pi + 400 - 50\pi - 200}{(4+\pi)^2}$$

$$= \frac{50\pi + 200}{(4+\pi)^2} = \frac{50(\pi+4)}{(\pi+4)^2}$$

$$\Rightarrow \quad A = \frac{50}{4+\pi} m^2 \qquad \text{...(7)}$$

(i) (a) The required relation is :

$2x + r(\pi + 2) = 10$ (from (2))

(ii) (c) Required area is :

$$A = 10r - \frac{\pi r^2}{2} - 2r^2 \qquad \text{(from (3))}$$

(iii) (d) Required value of r is :

$$r = \frac{10}{4+\pi} \qquad \text{(from (4))}$$

(iv) (b) Required dimensions of the window are :

$$2r = \frac{20}{\pi+4};\ x = \frac{10}{\pi+4} \qquad \text{(from (5), (6))}$$

(v) (c) Required maximum area of window $= \frac{50}{4+\pi}$ (from (7))

19. $\cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$

$$= \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}\right]$$

$$= \cot^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right]$$

$$= \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right] = \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2}$$

20. $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$

$\Rightarrow$ $1(2a^2 + 4) - 2(-4a - 20) + 0 = 86$

or $2a^2 + 8a + 44 = 86$

$\Rightarrow a^2 + 4a - 21 = 0$

or $(a + 7)(a - 3) = 0$

$\Rightarrow a = -7$ and 3

Hence; Sum $= -7 + 3 = -4$

OR

$|A| = 2 \neq 0$

$\Rightarrow \quad A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

Now ;

LHS $= 2A^{-1} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

and R.H.S. $= 9I - A = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

Hence; proved

21. $\lim_{x\to 0}\left(\frac{1-\cos 4x}{8x^2}\right) = \lim_{x\to 0}\left(\frac{2\sin^2 2x}{8x^2}\right)$

$$= \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 1$$

$\because \lim_{x\to 0} f(x) = f(0)$ [as f(x) is continuous at x = 0]

$\Rightarrow k = 1$

22. $y = x^3 - 11x + 5$ ...(1)

$\Rightarrow \dfrac{dy}{dx} = 3x^2 - 11$ = slope of tangent

Now, equation of tangent is $y = x - 11$

$\Rightarrow 3x^2 - 11 = 1$

or $x^2 = 4 \Rightarrow x = \pm 2$

From (1); $y = -9$ (at $x = 2$)

or $y = 19$ (at $x = -2$)

But $(-2, 19)$ does not satisfy the equation of tangent.

$\Rightarrow$ Required point is $(2, -9)$

23. Let $I = \int \dfrac{\sin x}{3+4\cos^2 x} dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\therefore\ I = -\int \frac{dt}{3+4t^2}$$

$$= \frac{-1}{4}\int \frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2 + t^2}$$

$$= \frac{-1}{4}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t}{\sqrt{3}}\right) + C$$

$$= \frac{-1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right) + C$$

OR

$$\int_2^5 \left[|x-2| + |x-3| + |x-5|\right] dx$$

$$= \int_2^3 [(x-2)-(x-3)-(x-5)]\,dx + \int_3^5 [(x-2)+(x-3)-(x-5)]\,dx$$

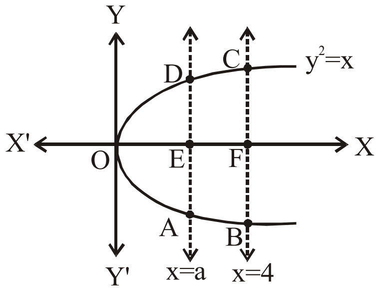
$$= \int_2^3 (-x+6)\,dx + \int_3^5 x\,dx$$

$$= \left[\frac{-x^2}{2} + 6x\right]_2^3 + \left[\frac{x^2}{2}\right]_3^5$$

$$= \left[\left(\frac{-9}{2}+18\right) - (-2+12)\right] + \frac{1}{2}(25-9)$$

$$= \frac{-9}{2} + 8 + 8 = \frac{23}{2}$$

24.



As per the given conditions;

Ar(OAD) = Ar(ABCD)

$\Rightarrow$ Ar(OED) = Ar(EFCD)

$\Rightarrow \int_0^a \sqrt{x}\,dx = \int_a^4 \sqrt{x}\,dx$

or $\left[\dfrac{x^{3/2}}{3/2}\right]_0^a = \left[\dfrac{x^{3/2}}{3/2}\right]_a^4$

$\Rightarrow a^{3/2} = 4^{3/2} - a^{3/2}$

or $2a^{3/2} = 8 \Rightarrow a = (4)^{2/3}$

25. $(1-y^2)(1+\log x)dx + 2xy\ dy = 0$

$\Rightarrow \dfrac{1+\log x}{x}dx = \dfrac{-2y}{1-y^2}dy$

On integrating both sides, we get

$\dfrac{(1+\log x)^2}{2} = \log|1-y^2| + C$

When x = 1, y = 0

$\Rightarrow \dfrac{(1+\log 1)^2}{2} = \log(1) + C \Rightarrow C = \dfrac{1}{2}$

hence; $(1 + \log x)^2 = 2\log|1 - y^2| + 1$ is the required solution.

26. $(\vec{a}+\vec{b}) = (2\hat{i} - 3\hat{j} + \hat{k}) + (-\hat{i} + \hat{k})$

$= \hat{i} - 3\hat{j} + 2\hat{k}$

and $(\vec{b}+\vec{c}) = (-\hat{i}+\hat{k}) + (2\hat{j} - \hat{k})$

$= -\hat{i} + 2\hat{j}$

$\Rightarrow (\vec{a}+\vec{b})\times(\vec{b}+\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{vmatrix} = -4\hat{i} - 2\hat{j} - \hat{k}$

$\therefore$ Required area of parallelogram

$$= \frac{1}{2}\left|(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})\right| = \frac{1}{2}\sqrt{(-4)^2+(-2)^2+(-1)^2}$$

$$= \frac{\sqrt{21}}{2} \text{ sq. units}$$

27. We have; $\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}$

($\because$ normal to plane is equally inclined to axes)

Hence; $\vec{N} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

$\Rightarrow$ Equation of plane is :

$\vec{r}.\hat{N} = 3\sqrt{3}$

$$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k}).\frac{\left(\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right)}{1} = 3\sqrt{3}$$

or $\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = 3\sqrt{3}$

$\Rightarrow$ Required equation of plane is :

$x + y + z = 9$

28. (i) $P(A'/B) = \frac{P(A'\cap B)}{P(B)}$

$$= \frac{P(B)-P(A\cap B)}{P(B)} = \frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{3}} = \frac{1}{4}$$

(ii) $P(A'/B') = \frac{P(A'\cap B')}{P(B')}$

$$= \frac{1-P(A\cup B)}{P(B')} = \frac{1-[P(A)+P(B)-P(A\cap B)]}{P(B')}$$

$$= \frac{1-\left[\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right]}{1-\frac{1}{3}} = \frac{1-\frac{14}{24}}{\frac{2}{3}} = \frac{5}{8}$$

OR

(i) $k + \frac{k}{2} + \frac{k}{4} + \frac{k}{8} = 1$

$\Rightarrow 8k + 4k + 2k + k = (1 \times 8)$

or $k = \dfrac{8}{15}$

(ii) $P(X \le 2) + P(X > 2)$

$= \left(k + \dfrac{k}{2} + \dfrac{k}{4}\right) + \left(\dfrac{k}{8}\right)$

$= \dfrac{7k}{4} + \dfrac{k}{8}$

$= \dfrac{14k + k}{8} = \dfrac{15k}{8}$

$= \dfrac{15}{8} \times \dfrac{8}{15} = 1$

29. (i) Reflexive :

We observe that $\dfrac{1}{2} \le \left(\dfrac{1}{2}\right)^3$ is not true.

$\therefore \left(\dfrac{1}{2}, \dfrac{1}{2}\right) \notin S$. So, S is not reflexive.

(ii) Symmetric :

We observe that $1 \le 3^3$ but $3 \not\le 1^3$

i.e. $(1, 3) \in S$ but $(3, 1) \notin S$

So, S is not symmetric

(iii) Transitive :

We observe that $10 \le 3^3$ and $3 \le 2^3$ but $10 \not\le 2^3$

i.e. $(10, 3) \in S$ and $(3, 2) \in S$ but $(10, 2) \notin S$

So, S is not transitive

$\therefore$ S is neither reflexive, nor symmetric, nor transitive.

30. We have ; $y = (\cos x)^x + (\sin x)^{1/x}$

$\Rightarrow \quad y = u + v$ ...(1)

Now; $u = (\cos x)^x \Rightarrow \log u = x.\log \cos x$

$\Rightarrow \dfrac{1}{u}\dfrac{du}{dx} = x.\dfrac{1}{\cos x}(-\sin x) + \log \cos x\ .1$

$\Rightarrow \dfrac{du}{dx} = (\cos x)^x[-x \tan x + \log \cos x]$ ...(2)

and $v = (\sin x)^{1/x} \Rightarrow \log v = 1/x.\log \sin x$

$\Rightarrow \dfrac{1}{v}\dfrac{dv}{dx} = \dfrac{1}{x}.\dfrac{1}{\sin x}(\cos x) + \log \sin x\left(\dfrac{-1}{x^2}\right)$

$\therefore \frac{dv}{dx} = (\sin x)^{1/x}\left[\frac{\cot x}{x} - \frac{\log \sin x}{x^2}\right]$ ...(3)

From eqn's (1), (2) and (3); we have

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= (\cos x)^x [\log \cos x - x \tan x] + (\sin x)^{1/x}\left[\frac{\cot x}{x} - \frac{\log \sin x}{x^2}\right]$$

31. We have;

$$f(x) = |x - 3| = \begin{cases} x - 3, & x \geq 3 \\ -(x-3), & x < 3 \end{cases}$$

and $f(3) = |3 - 3| = 0$

Now; $L\, f'(3) = \lim_{h \to 0} \frac{f(3-h) - f(3)}{(-h)}$

$$= \lim_{h \to 0} \frac{[-(3-h-3)] - 0}{(-h)}$$

$$= \lim_{h \to 0} \frac{h}{(-h)} = -1$$

and $R\, f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$

$$= \lim_{h \to 0} \frac{(3+h-3) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = 1$$

$\because\ L\, f'(3) \neq R\, f'(3)$

$\Rightarrow$ f(x) is not differentiable at x = 3

OR

Here;

$x = \cos t\,(3 - 2\cos^2 t)$ and $y = \sin t\,(3 - 2\sin^2 t)$

$$\Rightarrow \frac{dx}{dt} = -\sin t(3 - 2\cos^2 t) + \cos t(2 \times 2 \cos t \sin t)$$

$\therefore \frac{dx}{dt} = -3 \sin t + 6 \cos^2 t \sin t$ ...(1)

and $\frac{dy}{dt} = \cos t\,(3 - 2\sin^2 t) + \sin t(-2.\ 2 \sin t \cos t)$

$\therefore \dfrac{dy}{dt} = 3\cos t - 6\sin^2 t\cos t$ ...(2)

Hence,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{3\cos t[1-2\sin^2 t]}{3\sin t[-1+2\cos^2 t]} = \frac{3\cos t}{3\sin t} \times \frac{\cos 2t}{\cos 2t}$$

$$\Rightarrow \frac{dy}{dx} = \cot t \Rightarrow \frac{dy}{dx}\left(\text{at } t = \frac{\pi}{4}\right)$$

$$= \cot\frac{\pi}{4} = 1$$

32. We have; $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$\Rightarrow f'(x) = 12x^3 - 12x^2 - 24x$

$= 12x(x^2 - x - 2)$

$\therefore f'(x) = 12x(x+1)(x-2)$

Now; $f'(x) = 0$

$\Rightarrow 12x(x+1)(x-2) = 0$ or $x = -1, 0, 2$

Interval	Sign of f '(x)	Nature of function
$(-\infty, -1)$	$(-)(-)(-) < 0$	S. D.
$(-1, 0)$	$(-)(+)(-) > 0$	S. I.
$(0, 2)$	$(+)(+)(-) < 0$	S. D.
$(2, \infty)$	$(+)(+)(+) > 0$	S. I.

(a) f(x) is strictly increasing in $(-1, 0) \cup (2, \infty)$

(b) f(x) is strictly decreasing in $(-\infty, -1) \cup (0, 2)$

33. Let $I = \int \dfrac{2x^2+1}{x^2(x^2+4)}dx$

If $x^2 = y \Rightarrow \dfrac{2x^2+1}{x^2(x^2+4)} = \dfrac{2y+1}{y(y+4)}$

Then; $\dfrac{2y+1}{y(y+4)} = \dfrac{A}{y} + \dfrac{B}{(y+4)}$

$\therefore A = \dfrac{1}{4}$ and $B = \dfrac{7}{4}$

[$\because 2y + 1 = A(y + 4) + By$ and putting $y = 0, -4$]

$$\Rightarrow \frac{2y+1}{y(y+4)} = \frac{1}{4}\times\frac{1}{y}+\frac{7}{4}.\frac{1}{y+4}$$

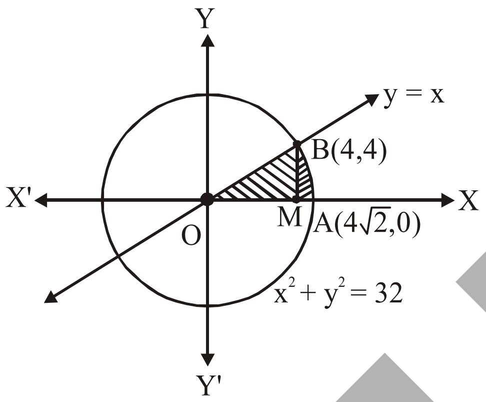
$$\therefore \frac{2x^2+1}{x^2(x^2+4)} = \frac{1}{4}\times\frac{1}{x^2}+\frac{7}{4}\times\frac{1}{x^2+4}$$

$$\Rightarrow \int \frac{2x^2+1}{x^2(x^2+4)}dx = \frac{1}{4}\int x^{-2}dx+\frac{7}{4}\int \frac{1}{x^2+2^2}dx$$

$$= \frac{1}{4}\times\frac{x^{-1}}{(-1)}+\frac{7}{4}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

$$= \frac{-1}{4x}+\frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right)+C$$

34. The given equations are : $y = x$ and $x^2 + y^2 = 32$



For intersection point of $y = x$ and $x^2 + y^2 = 32$;

$x^2 + x^2 = 32 \Rightarrow 2x^2 = 32$

or $x^2 = 16 \Rightarrow x = 4$; $y = x = 4$

Hence; Required Area = ar(region OBMO) + ar (region BMAB)

$$= \int_0^4 y\,dx + \int_4^{4\sqrt{2}} y\,dx$$

$$= \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{32-x^2}\,dx$$

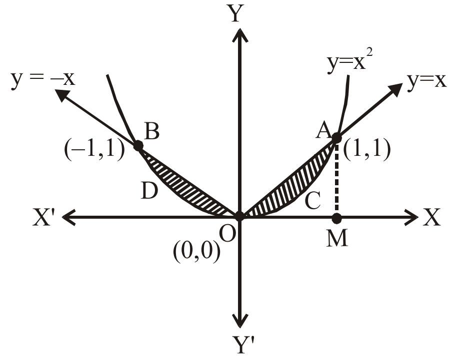
$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x}{2}\sqrt{32-x^2}+\frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left(\frac{1}{2}\times 4^2\right)+\left[\left(\frac{4\sqrt{2}}{2}\times 0+\frac{1}{2}\times 32\times\frac{\pi}{2}\right)-\left(\frac{4}{2}\sqrt{32-16}+\frac{1}{2}\times 32\times\frac{\pi}{4}\right)\right]$$

$= 8 + (8\pi - (8 + 4\pi))$

$= 4\pi$ sq. units

OR



The given equations are :

$y = x^2$ and $y = |x|$

For intersection points ; $x^2 = x$

$\Rightarrow x^2 - x = 0$ or $x(x-1) = 0$

$\Rightarrow x = 0,1$ or $y = 0,1$

Hence; required area

$$= 2\times\int_0^1 (y_2 - y_1)dx$$

$$= 2\times\int_0^1 (x - x^2)dx$$

$$= 2\times\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$$

$$= 2\times\left[\frac{1}{2}-\frac{1}{3}\right]$$

$$= 2\times\frac{1}{6} = \frac{1}{3}\text{sq. units}$$

35. We have; $(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}, |x| \neq 1$

$$\Rightarrow \frac{dy}{dx} + \frac{2xy}{x^2-1} = \frac{2}{(x^2-1)^2}$$

Comparing it with the linear differential equation of the form :

$\frac{dy}{dx} + Py = Q;$ where

$P = \frac{2x}{x^2-1}$ and $Q = \frac{2}{(x^2-1)^2}$

$\therefore$ Integrating factor

$= e^{\int P dx} = e^{\int \frac{2x}{x^2-1} dx}$

$= e^{\log_e (x^2-1)} = x^2 - 1$

Hence, solution of differential equation is :

$$y \times (x^2 - 1) = \int \frac{2}{(x^2-1)^2} \times (x^2-1)\, dx + C$$

$$\Rightarrow y(x^2 - 1) = 2\int \frac{dx}{x^2-1} + C$$

$$\text{or } y(x^2 - 1) = 2 \times \frac{1}{2} \log\left|\frac{x-1}{x+1}\right| + C$$

$$\Rightarrow y(x^2 - 1) = \log\left|\frac{x-1}{x+1}\right| + C$$

36. Here; $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix}$$

$= 1(-12+6) - 2(-8-6) - 3(-6-9)$

$= 67 \neq 0$

$\therefore A^{-1}$ exists

Now; $A_{11} = -6$, $A_{12} = 14$, $A_{13} = -15$

$A_{21} = 17$, $A_{22} = 5$, $A_{23} = 9$

$A_{31} = 13$, $A_{32} = -8$, $A_{33} = -1$

$$\therefore \text{adj } A = \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$$\text{Hence; } A^{-1} = \frac{\text{adj} A}{|A|} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

The given system of equations is :

$$x+2y-3z=-4$$
$$2x+3y+2z=2$$
$$3x-3y-4z=11$$

These system of equations can be written as :

AX = B; where

$$A=\begin{bmatrix}1 & 2 & -3\\ 2 & 3 & 2\\ 3 & -3 & -4\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, B=\begin{bmatrix}-4\\ 2\\ 11\end{bmatrix}$$

$\because$ A^{-1} exists; so system of equations has a unique solution given by $X = A^{-1}.B$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{67}\begin{bmatrix}-6 & 17 & 13\\ 14 & 5 & -8\\ -15 & 9 & -1\end{bmatrix}.\begin{bmatrix}-4\\ 2\\ 11\end{bmatrix}$$

$$=\frac{1}{67}\begin{bmatrix}201\\ -134\\ 67\end{bmatrix}=\begin{bmatrix}3\\ -2\\ 1\end{bmatrix}$$

$\Rightarrow x = 3, y = -2, z = 1$

OR

We have ;

$$\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}.\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}8 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 8\end{bmatrix}=8\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=8I$$

$$\Rightarrow \frac{1}{8}\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}.\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}=I$$

or $$\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}^{-1}=\frac{1}{8}\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}$$

Now, The given system of equations is :

$x-y+z = 4$,

$x-2y-2z = 9$,

and $2x+y+3z = 1$

These can be written as AX = B where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$\because$ A^{-1} exists $\Rightarrow X = A^{-1}.B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$\Rightarrow x = 3, y = -2, z = -1$

37. Required plane is given as $P_1 + \lambda P_2 = 0$

$\Rightarrow \quad \left[\vec{r}.(\hat{i} - 2\hat{j} + 3\hat{k}) - 4\right] + \lambda\left[\vec{r}.(-2\hat{i} + \hat{j} + \hat{k}) + 5\right] = 0$

$\Rightarrow \quad \vec{r}.\left[(1 - 2\lambda)\hat{i} + (-2 + \lambda)\hat{j} + (3 + \lambda)\hat{k}\right] = 4 - 5\lambda$ ....(1)

Now: x-axis intercept = y-axis intercept (given)

$\Rightarrow \quad \dfrac{4 - 5\lambda}{1 - 2\lambda} = \dfrac{4 - 5\lambda}{-2 + \lambda}$

$\therefore \quad 1 - 2\lambda = -2 + \lambda$

$\Rightarrow \quad 3\lambda = 3$

or $\quad \lambda = 1$, putting this value of λ in equation (1) ; we get

$\Rightarrow \quad \vec{r}.(-\hat{i} - \hat{j} + 4\hat{k}) = -1$

$\quad \vec{r}.(-\hat{i} - \hat{j} + 4\hat{k}) + 1 = 0$

OR

Equation of line passing through two given points (3, –4, –5) and (2, –3, 1) is

$$\frac{x - 3}{2 - 3} = \frac{y - (-4)}{-3 - (-4)} = \frac{z - (-5)}{1 - (-5)}$$

$\Rightarrow \quad \dfrac{x - 3}{-1} = \dfrac{y + 4}{1} = \dfrac{z + 5}{6}$ ....(1)

Now, equation of the plane passing through the points (1, 2, 3); (4, 2, –3) and (0, 4, 3) is

$$\begin{vmatrix} x - 1 & y - 2 & z - 3 \\ 4 - 1 & 2 - 2 & -3 - 3 \\ 0 - 1 & 4 - 2 & 3 - 3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x - 1 & y - 2 & z - 3 \\ 3 & 0 & -6 \\ -1 & 2 & 0 \end{vmatrix} = 0$$

$(x - 1)(0 + 12) - (y - 2)(0 - 6) + (z - 3)(6) = 0$

$\Rightarrow \quad 12x + 6y + 6z - 42 = 0$

$\Rightarrow \quad 2x + y + z - 7 = 0 \qquad$ ....(2)

From (1), we get

$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda$ (say)

$\Rightarrow$ A point on the line is given by

$x = -\lambda + 3;\ y = \lambda - 4;\ z = 6\lambda - 5$

This must satisfy the equation of the plane

$2(-\lambda + 3) + 1(\lambda - 4) + 6\lambda - 5 - 7 = 0 \qquad$ (from (2))

$\Rightarrow \quad 5\lambda - 10 = 0$

$\Rightarrow \quad \lambda = 2$

Hence, req. point is (1, –2, 7)

38. The given L.P.P is :

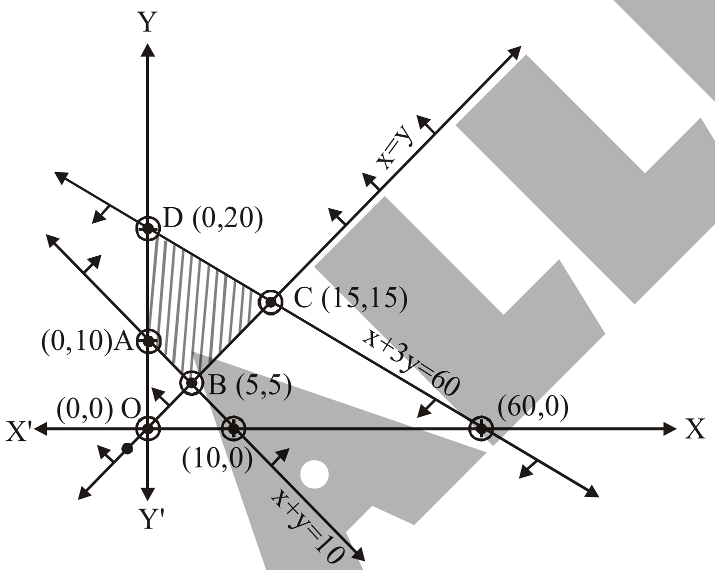
Minimise and Maximise

$Z = 3x + 9y$

subject to the constraints :

$x + 3y \le 60;\quad x + y \ge 10;\quad x \le y;\quad x \ge 0;\quad y \ge 0$

First of all, let us graph the given inequalities and find out the feasible region of the given L.P.P.



As shown in the figure, the given feasible region is bounded, shown by ABCD. And the corner points along–with the corresponding values of Z are shown in the table below :

Corner-points	Corresponding value of Z $Z = 3x + 9y$
A (0,10)	90
B (5,5)	60 (Minimum)
C (15,15)	180 (Maximum)
D (0,20)	180 (Maximum)

We can see that the problem has multiple optimal solutions at the corner points C and D i.e. both points produce same maximum value 180. Hence, it can be concluded that every point on the line-segment CD also gives the same maximum value. Hence, the minimum value of Z is 60 at the point B(5, 5) of the feasible region and the maximum value occurs at the two corner-points C(15, 15) and D(0, 20) and it is 180 in each case.

OR

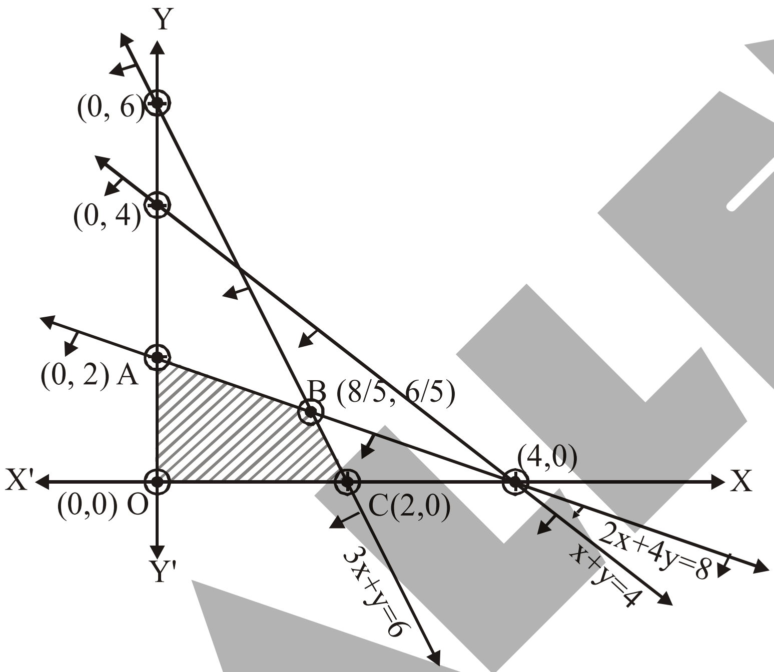
The given L.P.P. is :

Maximise $Z = 2x + 5y$

subject to the constraints

$2x + 4y \leq 8,\ 3x + y \leq 6,\ x + y \leq 4,\ x \geq 0,\ y \geq 0$

Firstly, let us represent the given inequalities graphically and find out the feasible region of the given L.P.P.



As shown in the figure; the given feasible region is bounded, shown by OABC. And the corner-points alongwith the corresponding values of Z are shown in the table below :

Corner-points	Corresponding value of Z $Z = 2x + 5y$
O (0,0)	0
A (0,2)	10 (Maximum)
B (8/5,6/5)	9.2
C (2,0)	4

Hence, the maximum value of Z is 10 at the point A (0, 2) of the feasible region.