REGIONAL MATHEMATICAL OLYMPIAD (RMO)–2019

(Held On Sunday 10th NOVEMBER , 2019)

Max. Marks : 102 **Time allowed : 3 hours**

TEST PAPER WITH SOLUTION

Instructions :

- **Calculators (in any form) and protactors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All questions carry equal marks. Maximum marks : 102.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. For each $n \in N$, let d_n denote the G.C.D. of n and (2019 – n). Find the value of $d_1 + d_2 + d_3 + \ldots + d_{2017} + d_{2018} + d_{2019}$.

Sol. Given $d_n = (n, 2019 - n)$

we know that, GCD (a, b) = GCD (a, b – a)

$\therefore$ GCD (n, 2019) = GCD (n, 2019 – n).

$\Rightarrow d_n = (2019, n)$

$\therefore \; d_1 + d_2 + d_3 + \ldots + d_{2019} = (2019, 1) + (2019, 2) + \ldots + (2019, 2019)$

As $2019 = 1 \times 3 \times 673$

$\therefore$ We have G. C. D of

(2019, multiple of 3) = 3 $\Rightarrow$ 672 possible number

(2019, multiple of 673) = 673 $\Rightarrow$ 2 possible number

(2019, 2019) = 2019 $\Rightarrow$ 1 possible number

(2019, remaining numbers) 1 $\Rightarrow$ 1344 possible number

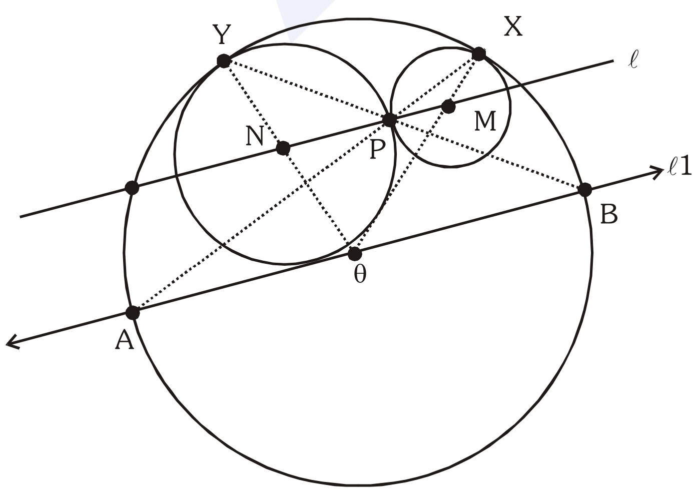
$\therefore \; d_1 + d_2 + \ldots + d_{2019}$

$= 1 \times 1344 + 672 \times 3 + 673 \times 2 + 2019 \times 1$

$= 6725$

2. Given a circle Γ, let P be a point in its interior, and let l be a line passing through P. Construct with proof using a ruler and compass, all circles which pass through P, are tangent to Γ, and whose center lies on line l.

Sol.



Construction :

(1) Draw a line ℓ^1 which is parallel to given line ℓ

(2) Let intersection point of ℓ^1 with given circle are A,B.

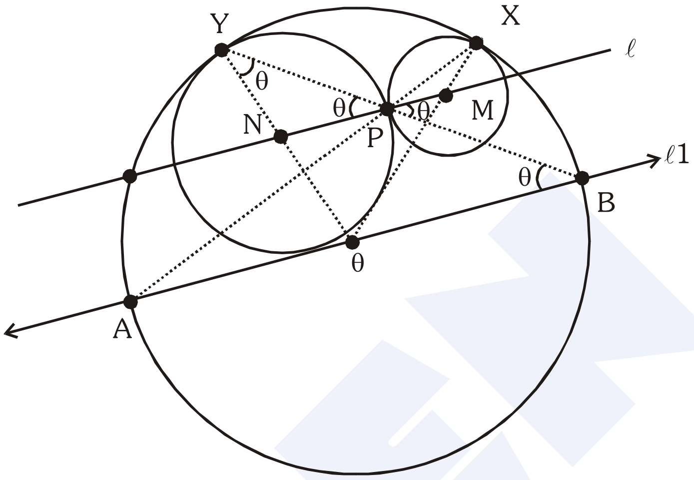
(3) Join $\overrightarrow{AP}$ and $\overrightarrow{BP}$ and extend to intersect the circle at X, Y.

(4) Join OX and OY. Let the intersection points of these line segments with given line 'ℓ' or M and N.

(5) Now draw the circles, with centres M, N and radius MP and NP.

There are the required circles.

JUSTIFICATION



Let $\angle OBY = \angle OYB = \theta$ (since OY = OB (Radii)

Since $\ell \parallel \ell'$ we have

$\angle YPN = \angle MPB = \angle MPO = \theta$

$\Rightarrow \angle NYP = \angle NPY = \theta$

$\Rightarrow$ NY = NP

The circle with radius NP touches the given circle at 'Y'.

Similarly the other circle also.

3. Find all triples of non-negative real numbers (a, b, c) which satisfy the following set of equations :

$a^2 + ab = c$

$b^2 + bc = a$

$c^2 + ca = b$

Sol. Given

$a^2 + ab = c \rightarrow a(a + b) = c$ ...(1)

$b^2 + bc = a \rightarrow b(b + c) = a$ ...(2)

$c^2 + ac = b \rightarrow c(a + c) = b$ ...(3)

Multiply eq^n (1), (2), (3)

$\Rightarrow abc(a + b)(b + c)(c + a) = abc$

$\Rightarrow (a+b)(b+c)(c+a) = 1$ ...(4)

Now,

$eq^n.(1) - eq^n.(2) \Rightarrow a^2 - b^2 = (c - a)(1 + b)$ ...(5)

$eq^n.(2) - eq^n.(3) \Rightarrow b^2 - c^2 = (a - b)(1 + c)$ ...(6)

$eq^n.(3) - eq^n.(1) \Rightarrow c^2 - a^2 = (b - c)(1 + a)$ ...(7)

Multiply eq^n. (5), (6), (7)

$\Rightarrow$ (a – b) (a + b) (b – c) (b + c) (c – a) (c + a) = (a – b) (b – c) (c – a) (1 + a) (1 + b) (1 + c)

$\Rightarrow$ (a – b) (b – c) (c – a) [1 – (1 + a) (1 + b) (1 + c)] = 0 $\quad$ ($\because$ (a + b) (b + c) (c + a) = 1)

This is possible if,

(a – b) (b – c) (c – a) = 0 $\quad$ or $\quad$ 1 = (1 + a) (1 + b) (1 + c)

Case 1 :

If (a – b) (b – c) (c – a) = 0 $\Rightarrow$ either a = b or b = c or c = a

Let a = b and using eq^n (1), (2), (3)

We get

$$a^2 + a^2 = c \Rightarrow \boxed{2a^2 = c} \quad \text{...(i)}$$

and $\left.\begin{matrix} a^2 + ac = a \\ c^2 + ac = a \end{matrix}\right]$ on subtraction we get

$$\Rightarrow a^2 = c^2$$

$$\Rightarrow \boxed{a = c} \quad \text{...(ii)}$$

using (i) and (ii)

$2a^2 = a \Rightarrow a = 0$ or $a = \dfrac{1}{2}$

Similarly, $b = 0$ or $b = \dfrac{1}{2}$

& $c = 0$ or $c = \dfrac{1}{2}$

$\therefore$ Possible triples (a, b, c) = (0, 0, 0)

$$= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

Case 2 :

(1 + a) (1 + b) (1 + c) = 1

Only possible when a = b= c = 0

which is already considered in above case.

Alternate solution

$a = \dfrac{c}{a+b}$, $b = \dfrac{a}{b+c}$, $c = \dfrac{b}{c+a}$

If any of a, b, c is zero, then other two will also becomes zero, so (0, 0, 0) is one trivial solution.

Now let $abc \neq 0$. So $a, b, c \in R^+$.

$$a + b + c = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

$$= \frac{a^2}{ab+ac} + \frac{b^2}{bc+ba} + \frac{c^2}{ca+cb}$$

$$\geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \qquad \text{(by Titu's inequality)}$$

$$\geq \frac{3(ab+bc+ca)}{2(ab+bc+ca)} \qquad \{\because (a+b+c)^2 \geq 3(ab+bc+ca)\}$$

$$\geq \frac{3}{2} \qquad \text{(it is also a famous nesbitt's inequality)}$$

So $a + b + c \geq \frac{3}{2}$ ....(i)

Also adding all given 3 equations,

we get $a^2 + b^2 + c^2 + ab + bc + ca = a + b + c$

$(a + b + c)^2 - 2(ab + bc + ca) + (ab + bc + ca) = a + b + c$

$(a + b + c)^2 - (a + b + c) = ab + bc + ca \leq \frac{(a+b+c)^2}{3}$

$\Rightarrow \frac{2}{3}(a + b + c)^2 \leq (a + b + c)$

$a + b + c \leq \frac{3}{2}$ ....(ii)

from (i) and (ii), we get $a + b + c = \frac{3}{2}$,

for which equality holds at $a = b = c = \frac{1}{2}$

so solution is $(0, 0, 0)$ $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

4. Let $a_1, a_2, ...a_6, a_7$ be seven positive integers. Let S be the set of all numbers of the form $a_i^2 + a_j^2$ where $1 \leq i < j \leq 7$. Prove that there exist two elements of S which have the same remainder on dividing by 36.

Sol. $a_1, a_2, ..., a_7 \in I^+$

No. of ways of selecting two numbers (a_i, a_j) out of 7 numbers is ${}^7C_2 = 21$

Now, let's check the divisibility of any perfect square (mod 36).

$1^2 = 1 \pmod{36}$

$2^2 = 4 \pmod{36}$

$3^2 = 9 \pmod{36}$

$4^2 = 16 \pmod{36}$

$5^2 = 25 \pmod{36}$

$6^2 = 0 \pmod{36}$

$7^2 = 13 \pmod{36}$

$8^2 = 28 \pmod{36}$

$9^2 = 9 \pmod{36}$

$10^2 = 28 \pmod{36}$

so any $n^2 \equiv 0, 1, 4, 9, 13, 16, 25, 28, \pmod{36}$

Case 1 : $a_i \equiv a_j \pmod{36}$

then we are directly done as both number gives same remainder when divided by 36

Case 2 : $a_i \neq a_j \pmod{36}$

then $a_i^2 + a_j^2 \equiv 1, 4, 13, 16, 25, 28, 5, 10, 14, 17, 26, 29, 20, 32, 22, 34, 2, 8 \pmod{36}$

These are 19 possible remainders since we have 21 pairs, so by PHP. Again we are done.

5. There is a pack of 27 distinct cards, and each card has three values on it. The first value is a shape from $(\Delta, \square, \odot)$; the second value is a letter from {A, B, C}; and the third value is a number from {1, 2, 3}. In how many ways can we choose an unordered set of 3 cards from the pack, so that no two of the chosen cards have two matching values?

For example, we can choose {ΔA1, ΔB2, $\odot$B3}. But we cannot choose {ΔA1,$\square$B2,ΔC1}

Sol. Let as select any card out of 27 distinct card this can be done 27 ways.

After first card we can't select 6 cards which will be having excatly two scripts common wth first card.

For second card there are two kinds of cards available nothing common with first card $2 \times 2 \times 2 = 8$ such cards. Exactly one script common with first card ${}^3C_1 \times 2 \times 2 = 12$ such card

Case (1)

Second card selected from nothing common with first then for third card we will have $27 - 7 - 7 = 13$ options

$$\Rightarrow \text{ No of ways } = \frac{27 \times 8 \times 13}{3!} = 9 \times 4 \times 13 = 468$$

Case (2)

Second card selected from one script common with first selected card then for third card we will be having $27 - 9 + 1 - 4 = 15$ ways

$$\text{No. of ways } = \frac{27 \times 12 \times 15}{3!} = 810$$

Total ways = 810 + 468 = 1278 ways

6. Let k be a positive real number. In the X–Y coordinate plane let S be the set of all points of the form $(x, x^2 + k)$ where $x \in R$. Let C be the set of all circles whose center lies in S, and which are tangent to the X-axis. Find the minimum value of k such that any two circles in C have at least one point of intersection.

Sol. Note that the locus of S is parabola i.e. $y = x^2 + k$

with focus (0, K) and latus rectum = 1 (4a = 1)

Now circle with centres $(x_1, x_1^2 + K)$, has radius $x_1^2 + K$ as these circles are touching X-axis.

Let two circles with centres

$(x_1, x_1^2 + K)$ and $(x_2, x_2^2 + K)$ with radii $x_1^2 + K$ and $x_2^2 + K$ respectively.

Now if two circles should intersect at least one point

$\Rightarrow$ distances between centres $\le$ sum of radii

$\Rightarrow$ $(x_1 - x_2)^2 + (x_1^2 + K - x_2^2 - K)^2 \le (x_2^2 + K + x_2^2 + K)^2$ ....(1)

$\Rightarrow$ $(x_1 - x_2)^2 + (x_1^2 - x_2^2)^2 \le (x_1^2 + x_2^2 + 2K)^2$

$(x_1 - x_2)^2 [1 + (x_1 + x_2)^2] \le (x_1^2 + x_2^2 + 2K)^2$

$\Rightarrow$ we have $(x_1^2 + x_2^2 + 2K)^2 \ge (x_1 - x_2)^2$

$x_1^2 + x_2^2 + 2K \ge |x_1 - x_2|$

for strong inequality we will take

x_1 and x_2 as opposite signs

$\Rightarrow$ $x_1^2 + x_2^2 + 2K \ge |x_1| + |x_2|$

$2K \ge |x_1| - x_1^2 + |x_2| - x_2^2$

$2K \ge |x_1| - |x_1|^2 + |x_2| - |x_2|^2$

maximum is achieved at $x_1 = x_2 = \dfrac{1}{2}$

$\therefore$ $2K \ge \dfrac{1}{2}$

$\Rightarrow$ $K \ge \dfrac{1}{4}$

Now we can check eq^n (1) inequality is true for this $K = \dfrac{1}{4}$ value

$$\left(x_1^2 + x_2^2 + \frac{1}{2}\right)^2 \ge (x_1 - x_2)^2 + (x_1^2 - x_2^2)^2$$

after simplifying and applying AM-GM, is true

$\therefore$ the required minimum value of $K = \dfrac{1}{4}$