REGIONAL MATHEMATICAL OLYMPIAD (RMO)–2019

(Held On Sunday 20th OCTOBER , 2019)

Max. Marks : 102 **Time allowed : 3 hours**

TEST PAPER WITH SOLUTION

Instructions :

- **Calculators (in any form) and protactors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All questions carry equal marks. Maximum marks : 102.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Suppose x is a nonzero real number such that both x^5 and $20x + \frac{19}{x}$ are rational numbers. Prove that x is a rational number.

Sol. Let $20x + \dfrac{19}{x} = r,\ r \in \mathbb{Q}$

$\Rightarrow\ 20x^2 - rx + 19 = 0$

$\Rightarrow\ x = \dfrac{r \pm \sqrt{r^2 - 4\cdot 20\cdot 19}}{40}$

$\Rightarrow\ x = r_1 \pm \sqrt{r_2}\ ,\ \ r_1, r_2 \in \mathbb{Q}\ ;$ as $x \in \mathbb{R}$ $\Rightarrow$ $r_2 \in \mathbb{R} \geq 0$

Now $x^5 = (r_1 \pm \sqrt{r_2})^5 = \left(\binom{5}{0}r_1^5 + \binom{5}{2}r_1^3 r_2 + \binom{5}{4}r_1 r_2^2\right) \pm\sqrt{r_2}\left(\binom{5}{1}r_1^4 + \binom{5}{3}r_1^2 r_2 + \binom{5}{5}r_2^2\right)$

as $x^5 \in \mathbb{Q}$ and $5r_1^4 + 10r_1^2 r_2 + r_2^2 \neq 0$

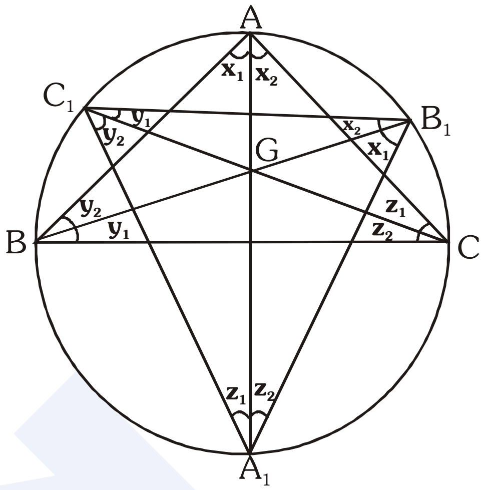
otherwise $r_1 = r_2 = 0 \Rightarrow x = 0$ (not possible)

$\Rightarrow\ \sqrt{r_2} \in \mathbb{Q}$

$\Rightarrow\ x \in \mathbb{Q}$, Hence proved.

2. Let ABC be a triangle with circumcircle Ω and let G be the centroid of triangle ABC. Extend AG, BG and CG to meet the circle Ω again in A_1, B_1 and C_1, respectively. Suppose $\angle BAC = \angle A_1B_1C_1$, $\angle ABC = \angle A_1C_1B_1$ and $\angle ACB = \angle B_1A_1C_1$. Prove that ABC and $A_1B_1C_1$ are equilateral triangles.

Sol. (As equal chords makes equal angles)



Given that $\angle BAC = \angle A_1B_1C_1 \Rightarrow BC = A_1C_1$

$\angle ABC = \angle A_1C_1B_1 \Rightarrow A_1B_1 = AC$

$\angle ACB = \angle B_1A_1C_1 \Rightarrow B_1C_1 = AB$

Since A, B, C, A_1,B_1,C_1 all are cyclic

We have $\angle x_2 = \angle y_2$ and $\angle y_2 = \angle z_2$

$\Rightarrow \angle x_2 = \angle z_2$

Now in triangle ΔBGC, we have

$\angle BGC = 180 - (y_1 + z_2)$ ...(1)

But In ΔABC we have

$x_1 + x_2 + y_1 + y_2 + z_1 + z_2 = 180^\circ$

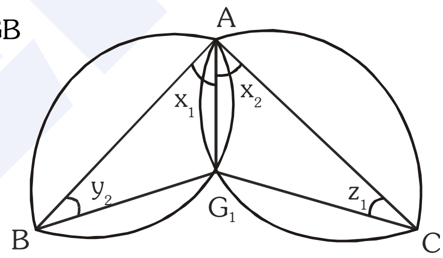
$\therefore \angle y_1 + \angle z_2 = 180 - (x_1 + x_2 + y_2 + z_1)$

$\therefore \angle BGC = x_1 + x_2 + y_2 + z_1$

But we got $x_2 = y_2$ (alternate angles)

and AG is radical axis and AC is tangent to circumcircle of ΔAGB

Similarly AB is tangent to circumcircle of ΔAGC



$\therefore x_1 = z_1$

$\therefore \angle BGC = 2(x_1 + x_2) = 2\angle A$

So G is circumcentre of ΔABC

since G is given as centroid

$\therefore \Delta ABC$ is an equilateral triangle

& $\Delta A_1B_1C_1$ is also an equilateral triangle. Hence proved.

3. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{a}{a^2+b^3+c^3} + \frac{b}{b^2+c^3+a^3} + \frac{c}{c^2+a^3+b^3} \le \frac{1}{5abc}.$$

Sol. $a^2 + b^3 + c^3 = a^2 \cdot 1 + b^3 + c^3$

$= a^2(a+b+c) + b^3 + c^3$

$= a^3 + b^3 + c^3 + a^2b + a^2c$

$\ge 5\,(a^7 b^4 c^4)^{1/5}$ (By AM $\ge$ GM)

$$\Rightarrow \frac{1}{a^2+b^3+c^3} \le \frac{1}{5(a^7.b^4c^4)^{\frac{1}{5}}}$$

$$\Rightarrow \frac{a}{a^2+b^3+c^3} \le \frac{(a^3bc)^{\frac{1}{5}}}{5abc} \le \frac{\frac{3a+b+c}{5}}{5abc} \quad \text{(By AM} \ge \text{GM)}$$

$$\Rightarrow \frac{a}{a^2+b^3+c^3} \le \frac{1}{5abc}\left(\frac{3a+b+c}{5}\right)$$

$$\Rightarrow \text{ LHS} \le \frac{1}{5abc}\left(\sum\left(\frac{3a+b+c}{5}\right)\right)$$

$$\le \frac{1}{5abc}\frac{5(a+b+c)}{5}$$

$$\le \frac{1}{5abc} \quad (\text{as} \quad a+b+c=1)$$

Hence proved.

4. Consider the following 3×2 array formed by using the numbers 1, 2, 3, 4, 5, 6 :

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}$$

Observe that all row sums are equal, but the sum of the squares is not the same for each. Extend the above array to a $3 \times k$ array $(a_{ij})_{3 \times k}$ for a suitable k, adding more columns, using the numbers 7, 8, 9,....., 3k such that

$$\sum_{j=1}^{k} a_{1j} = \sum_{j=1}^{k} a_{2j} = \sum_{j=1}^{k} a_{3j} \quad \text{and} \quad \sum_{j=1}^{k} (a_{1j})^2 = \sum_{j=1}^{k} (a_{2j})^2 = \sum_{j=1}^{k} (a_{3j})^2$$

Sol. $1 + 2 + 3 + \ldots + 3k = \dfrac{3k(3k+1)}{2}$

and $\displaystyle\sum_{j=1}^{k} a_{1j} = \frac{k(3k+1)}{2} = \text{integer}$

Also $1^2 + 2^2 + 3^2 + \ldots + (3k)^2 = \dfrac{3k(3k+1)(6k+1)}{6}$

As $\displaystyle\sum_{j=1}^{k} a_{1j}^2 = \frac{1}{3}\frac{k(3k+1)(6k+1)}{2} \Rightarrow 3 \mid k$

Claim ; If $3 \mid k$ and $k > 3$ then it is always possible.

Proof : Observe following :

$(n^2 + (n+5)^2) - ((n+1)^2 + (n+4)^2) = 8$

$((m+1)^2 + (m+4)^2 - ((m+2)^2 + (m+3)^2) = 4$

$(\ell^2 + (\ell+5)^2) - ((\ell+2)^2 + (\ell+3)^2) = 12$

Also $8 + 4 = 12$

$\Rightarrow (n^2 + (n+5)^2) + (m+1)^2 + (m+4)^2 + (\ell+2)^2 + (\ell+3)^2$

$= (n+1)^2 + (n+4)^2 + (m+2)^2 + (m+3)^2 + \ell^2 + (\ell+5)^2$

Also $\quad n + (n+5) + (m+1) + (m+4) + (\ell+2) + (\ell+3)$

$= 2n + 2m + 2\ell + 15$

$= (n+1) + (n+4) + (m+2) + (m+3) + (\ell) + (\ell+5)$

Hence $\quad 1^2 + 6^2 + 8^2 + 11^2 + 15^2 + 16^2$

$2^2 + 5^2 + 9^2 + 10^2 + 13^2 + 18^2$

$3^2 + 4^2 + 7^2 + 12^2 + 14^2 + 17^2$

$$\Rightarrow \begin{pmatrix} 1 & 6 & 8 & 11 & 15 & 16 \\ 2 & 5 & 9 & 10 & 13 & 18 \\ 3 & 4 & 7 & 12 & 14 & 17 \end{pmatrix}$$

is satisfying the desired condition.
Similarly we can find

$$\begin{pmatrix} 1 & 6 & 8 & 11 & 18 & 13 & 21 & 23 & 25 \\ 2 & 5 & 7 & 12 & 15 & 17 & 19 & 22 & 27 \\ 3 & 4 & 9 & 10 & 14 & 16 & 20 & 24 & 26 \end{pmatrix}$$ which is satisfying all condition

Also observe $(n + 1)^2 + (n + 6)^2 + (n + 8)^2 + (n + 11)^2 + (n + 15)^2 + (n + 16)^2$
$= (n + 2)^2 + (n + 5)^2 + (n + 9)^2 + (n + 10)^2 + (n + 13)^2 + (n + 18)^2$
$= (n + 3)^2 + (n + 4)^2 + (n + 7)^2 + (n + 12)^2 + (n + 14)^2 + (n + 17)^2$

Hence we can always get a construction for $k + 6$ from a construction of k as we already got for $k = 6$ and $k = 9$, by induction it is true for all k such that $3|k, k > 3$

for $k = 3$ we can directly check that it will not happen. done!!

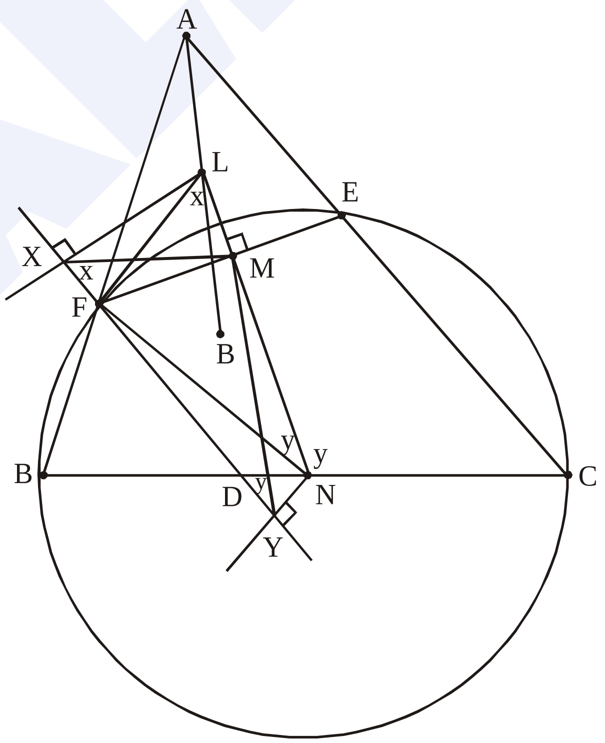
5. In an acute angled triangle ABC, let H be the orthocenter, and let D, E, F be the feet of altitudes from A, B, C to the opposite sides, respectively. Let L, M, N be midpoints of segments AH, EF, BC, respectively. Let X, Y be feet of altitudes from L, N on to the line DF. Prove that XM is perpendicular to MY.

Sol. Since $BE \perp AC$ and $CF \perp AB$.

So we have AFHE is cyclic quadrilateral also AH is diameter of this circle since L is the mid point of AH the EF is chord of circle we have $LM \perp EF$ (as M is mid point of EF)

Similarly we have BCEF is also cyclic and 'N' is mid point of diameter of BC.

Since EF is radical axis of both the circle $\odot$AFHE and $\odot$BCEF and L, N are centres of these circles, so $LN \perp EF$.



$\therefore$ We have LMN are colinear.

Now LMFX, MF YN are cyclic (in both quadrilaterals sum of opposite angles = 180°)

$\therefore$ $\angle MLF = \angle MYF = x$ (Let)

$\angle MNF = \angle MYF = y$ (Let)

$\therefore$ $\angle XMY = \angle LFN = 90°$ (as $x + y = 90°$)

Since $\angle LFN$ is angle in the semicircle of nine point circle with LN as diameter.

$\therefore$ $XM \perp MY$ proved

6. Suppose 91 distinct positive integers greater than 1 are given such that there are at least 456 pairs among them which are relatively prime. Show that one can find four integers a, b, c, d among them such that gcd(a, b) = gcd(b, c) = gcd(c, d) = gcd(d, a) = 1.

Sol. Let us consider a graph G with 91 vertices (91 distinct numbers) and connect each pair of these vertices which corresponds to co-prime pairs $\Rightarrow$ at least 456 edges i.e. $e \geq 456$. Here e = number of edges. Now getting four number a,b, c, d such that gcd (a, b) = gcd (b, c) = gcd(c,d) = gcd (d,a) = 1 $\Rightarrow$ existence at a cycle of length 4.

Let us assume there is no cycle of length '4'.

Let vertex set of G be $V = \{v_1, v_2, \ldots, v_{91}\}$, let deger of $v_i = d_i$.

For any vertex $v_i \in V$, the number of vertex pairs (v_α, v_β) adjacent to v_i is $\binom{d_i}{2}$.

As G contain no cycle of length 4, when v_i is changing in the V, all vertex pairs (v_α, v_β) are distinct. Otherwise,

vertex pairs (v_α, v_β) are counted in both $\binom{d_i}{2}$ and $\binom{d_j}{2}$ respectively.

Then $v_i, v_\alpha, v_j, v_\beta$ form a cycle of length 4.

$$\Rightarrow \sum_{i=1}^{91}\binom{d_i}{2} \leq \binom{91}{2}$$

$$\Rightarrow \sum_{i=1}^{91}\left(\frac{d_i^2}{2} - \frac{d_i}{2}\right) \leq 91 \times 45$$

$$\Rightarrow \frac{1}{2}\sum_{i=1}^{91} d_i^2 - e \leq 91 \times 45 \quad (\text{as } \Sigma d_i = 2e) \qquad \ldots(1)$$

$$\text{Also } \left(\frac{\sum_{i=1}^{91} d_i^2}{91}\right)^{1/2} \geq \left(\frac{\sum_{i=1}^{91} d_i}{91}\right) = \frac{2e}{91}$$

$$\Rightarrow \sum_{i=1}^{91} d_i^2 \geq \frac{4e^2}{91} \qquad \ldots(2)$$

From (1) and (2), $\frac{2e^2}{91} - e \leq 91 \times 45$

$\Rightarrow$ $(4e)^2 - 2.\ 4.e.\ 91 + 91^2 \leq 360 \cdot 91^2 + 91^2$

$\Rightarrow$ $(4e - 91)^2 \leq 91^2(361)$

$\Rightarrow$ $4e \leq 91 + 91 \times 19$

$\Rightarrow$ $4e \leq 91 \times 20$

$\Rightarrow$ $e \leq 91 \times 5 = 455$

But $e \geq 456$ contradiction!!

Hence there must exist a cycle of length '4'.