PRE-REGIONAL MATHEMATICAL OLYMPIAD (PRMO)–2019 FINAL EXAMINATION

(Held On Sunday 11th AUGUST, 2019)

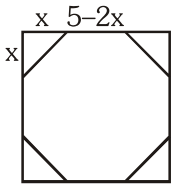
Max. Marks : 102 **Time allowed : 3 hours**

TEST PAPER WITH SOLUTION

1. From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Find the area removed to the nearest integer ?

Ans. 4

Sol.



So side of octagon = $5 - 2x = \sqrt{2}x$

$\Rightarrow \quad x = \dfrac{5}{2+\sqrt{2}}$

$\therefore$ Area removed $= 4\left(\dfrac{x^2}{2}\right)$

$= 2(x^2) \cong 4.29 \cong 4$

2. Let $f(x) = x^2 + ax + b$. If for all nonzero real x

$$f\left(x + \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

and the roots of $f(x) = 0$ are integers, what is the value of $a^2 + b^2$?

Ans. 13

Sol. $f(x) = x^2 + ax + b$

$f\left(x+\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$

$\Rightarrow \left(x+\dfrac{1}{x}\right)^2 + a\left(x+\dfrac{1}{x}\right) + b = x^2 + ax + b + \left(\dfrac{1}{x}\right)^2 + \dfrac{a}{x} + b$

$\Rightarrow \boxed{b = 2}$

$\therefore \ f(x) = x^2 + ax + 2$

for integer roots, D is a perfect square

$\Rightarrow a^2 - 8 = K^2$

$\Rightarrow (a - K)(a + K) = 8$

$\therefore \ a = 3, K = 1$

Hence $a^2 + b^2 = 13$

3. Let x_1 be a positve real number and for every integer $n \geq 1$ let $x_{n+1} = 1 + x_1x_2...x_{n-1}x_n$. If $x_5 = 43$, what is the sum of digits of the largest prime factor of x_6 ?

Ans. 13

Sol. $x_{n+1} = 1 + x_1x_2....x_n$

$x_2 = 1 + x_1$

$\vdots$

$x_5 = 1 + x_1x_2x_3x_4 = 43$ (given)

$\Rightarrow \boxed{x_1x_2x_3x_4 = 42}$

Now, $x_6 = 1 + (x_1x_2x_3x_4)x_5$

$= 1 + (42)(43) = 1807$

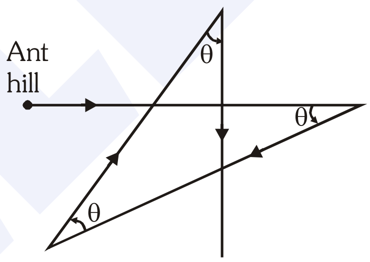
$\therefore 1807 = 13 \times 139 \rightarrow$ largest prime factor

$\therefore$ Sum of digits of the largest prime factor x_6 is $1 + 3 + 9 = 13$

4. An ant leaves the anthill for its morning exercise. It walks 4 feet east and then makes a 160° turn to the right and walks 4 more feet. It then makes another 160° turn to the right and walks 4 more feet. If the ant continues this pattern until it reaches the anthill again, what is the distance in feet it would have walked ?

Ans. 36

Sol. $\theta = \frac{\pi}{9} = 20°$



Ant is moving in a star pattern.

Let there be n sides.

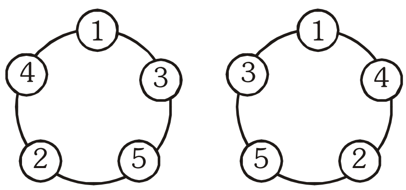
So $n\left(\frac{\pi}{9}\right) = \pi \Rightarrow n = 9$

Total distance covered $= 9 \times 4 = 36$

5. Five persons wearing badges with numbers 1, 2, 3, 4, 5 are seated on 5 chairs around a circular table. In how many ways can they be seated so that no two persons whose badges have consecutive numbers are seated next to each other ? (Two arrangements obtained by rotation around the table are considered different.)

Ans. 10

Sol.



Because the seats are distinct.

Total possible arrangements are $2 \times 5 = 10$

6. Let $\overline{abc}$ be a three digit number with nonzero digits such that $a^2 + b^2 = c^2$. What is the largest possible prime factor of $\overline{abc}$?

Ans. 29

Sol. $a^2 + b^2 = c^2$ (a,b,c are unit digit no's and $a \neq 0$)

$\rightarrow$ only possible at

$a = 3,\ b = 4,\ c = 5$

or

$a = 4,\ b = 3,\ c = 5$

So the no's are $\rightarrow 345 = 3 \times 5 \times 23$

or

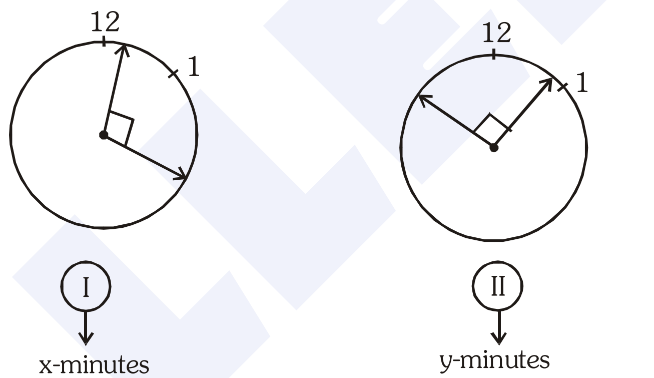
$435 = 3 \times 5 \times 29$

Largest possible prime factor = 29

7. On a clock, there are two instants between 12 noon and 1 PM, when the hour hand and the minute hand are at right angles. The difference in minutes between these two instants is written as $a + \frac{b}{c}$, where a, b, c are positive integers, with b < c and $\frac{b}{c}$ in the reduced form. What is the value of a + b + c ?

Ans. 51

Sol.



1 minute = 6°

$$6x - \left(\frac{x}{60}\right)30 = 90 \qquad 6y - \left(\frac{y}{60}\right)30 = 270$$

$$x = \frac{180}{11} \qquad y = \frac{540}{11}$$

$$y - x = \frac{540-180}{11} = \frac{360}{11} = 32 + \frac{8}{11}$$

$$= a + \frac{b}{c}$$

$\therefore\ a + b + c = 51.$

8. How many positive integers n are there such that $3 \le n \le 100$ and $x^{2^n} + x + 1$ is divisible by $x^2 + x + 1$?

Ans. 49

Sol. $x^2 + x + 1 \mid x^{2^n} + x + 1$

Observe that roots of $x^2 + x + 1$ one ω and ω^2.

i.e. if $P(x) = x^{2^n} + x + 1$

then $P(\omega) \Big| \; P(\omega^2) = 0$

$\omega^{2^n} + \omega + 1 = 0$ ...(i)

also, we know that $1 + \omega + \omega^2 = 0$ ...(ii)

$\therefore$ Compairing eq^n. (i) and (ii), we get

$$\omega^{2^n} = \omega^2$$

also, $\omega^3 = 1$, i.e. 2^n must be of the form $3K + 2$, so as to make $\omega^{2^n} = \omega^2$.

Dividing 2^n by 3, we get

$2^n = (-1)^n \pmod 3$.

Clearly for $x \in$ odd. $2^n = 2 \pmod 3$

$\therefore$ for $3 \le n \le 100$

there are 49 possible value of n.

9. Let the rational number p/q be closest to but not equal to 22/7 among all rational numbers with denominator < 100. What is the value of p – 3q ?

Ans. 14

Sol. $\frac{p}{q} \approx \frac{22}{7}$

$\Rightarrow \frac{p}{q} \approx \frac{308}{98}$

But $\frac{p}{q} \neq \frac{308}{98}$

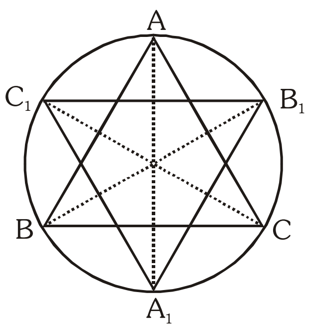
$\therefore \frac{p}{q} = \frac{311}{99}$

$\Rightarrow p - 3q = 14$

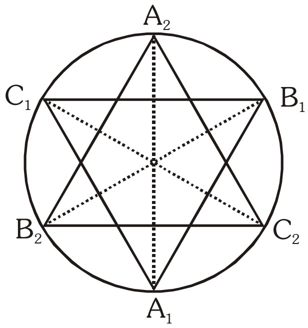
10. Let ABC be a triangle and let Ω be its circumcircle. The internal bisectors of angles A, B and C intersect Ω at A_1, B_1 and C_1, respectively, and the internal bisectors of angles A_1, B_1 and C_1 of the triangle $A_1B_1C_1$ intersect Ω at A_2, B_2 and C_2, respectively. If the smallest angle of triangle ABC is 40°, what is the magnitude of the smallest angle of triangle $A_2B_2C_2$ in degrees ?

Ans. 55

Sol. In ΔABC,



Again, In $\Delta A_1B_1C_1$



Now, angles in $\Delta A_2B_2C_2$ are

$$\angle B_2A_2C_2 = \frac{\angle B + \angle C}{4} + 20° = \frac{140°}{4} + 20° = 55°$$

$$\text{Also, } \angle A_2B_2C_2 = \frac{\angle B + \angle C}{4} + 10° + \frac{\angle B}{4} + 10°$$

$$\text{and } \angle A_2B_2C_2 = \frac{\angle B + \angle C}{4} + \frac{\angle C}{4} + 10°$$

$\therefore$ Smallest angle = 55°

11. How many distinct triangles ABC are there, up to similarity, such that the magnitudes of angles A, B and C in degrees are positive integers and satisfy

$$\cos A \cos B + \sin A \sin B \sin kC = 1$$

for some positive integer k, where kC does not exceed 360° ?

Ans. 6

Sol. $\cos A \cos B + \sin A \sin B \sin kC = 1$

Multiply both sides by 2.

$2 \cos A \cos B + 2\sin A \sin B \sin kC = 2$

$2 \cos A \cos B + 2\sin A \sin B \sin kC = 1 + 1$

$2 \cos A \cos B + 2\sin A \sin B \sin kC = \sin^2A + \cos^2A + \sin^2B + \cos^2B$

$2\cos A \cos B + 2\sin A \sin B \sin kC = (\cos^2A + \cos^2B) + (\sin^2A + \sin^2B)$

$\Rightarrow \quad (\cos^2A + \cos^2B - 2\cos A \cos B) + (\sin^2A + \sin^2B - 2\sin A \sin B) + 2\sin A \sin B - 2\sin A \sin B \sin kC = 0$

$\Rightarrow \quad (\cos A - \cos B)^2 + (\sin A - \sin B)^2 + 2\sin A \sin B (1 - \sin kC) = 0$

Now A, B, C are the angles of a triangle.

So all must less than π.

The first two terms are perfect squares so always greater than or equal to zero.

Third term is positive as sine of A & B are positive and $(1 - \sin kC)$ too will be positive as $\sin kC < 1$.

$\therefore$ This equation is possible only when all three terms becomes zero.

$$\therefore \left.\begin{aligned} \cos A - \cos B = 0 \quad &\Rightarrow \cos A = \cos B \\ \sin A - \sin B = 0 \quad &\Rightarrow \sin A = \sin B \end{aligned}\right\} A = B$$

$\sin A > 0$

$\sin B > 0$

$\therefore \quad 1 - \sin kC = 0$

$\sin kC = 1$

since kC does not exceeds 360°

$\therefore \quad kC \le 360°$

so kC can be 90° only.

kC = 90

k = 1, C = 90 Also k = 90, C = 1

k = 2, C = 45 k = 45, C = 2

$k = 3,\ C = 30$	$k = 30,\ C = 3$
$k = 5,\ C = 18$	$k = 18,\ C = 5$
$k = 6,\ C = 15$	$k = 15,\ C = 6$
$k = 10\ \ C = 9$	$k = 9,\ \ C = 10$

$\because$ B = A

So 2A + C = 180

So C has to be even integer, then only A will be even.

So out of 12 cases, choose those cases where C is even hence total 6 triangles are possible.

12. A natural number $k > 1$ is called good if there exist natural numbers

$$a_1 < a_2 < \ldots < a_k$$

such that

$$\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \ldots + \frac{1}{\sqrt{a_k}} = 1.$$

Let f(n) be the sum of the first n good numbers, $n \geq 1$. Find the sum of all values of n for which $f(n + 5)/f(n)$ is an integer.

Ans. **18**

Sol. At first we observe that a_i's must be perfect square.

Now if we write a number as a sum of its distinct divisors, then we get a good number.

e.g. $6 = 1 + 2 + 3$

$$\Rightarrow \quad \frac{1}{6} + \frac{1}{3} + \frac{1}{2} = 1$$

$$\Rightarrow \quad \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{36}} = 1$$

$\Rightarrow$ 3 is a good number.

Similarly, $12 = 1 + 2 + 3 + 6$ $\Rightarrow$ 4 is a good number.

Also, $24 = 1 + 2 + 3 + 6 + 12$ $\Rightarrow$ 5 is a good number.

Also, $24 = 1 + 2 + 3 + 6 + 4 + 8$ $\Rightarrow$ 6 is a good number.

So, we can find a number that can be written as a sum of its K distinct divisors. So, K is a good number $\forall\, K \geq 3$.

Now, $K \neq 2$

(as if $a + b = n$ & $\frac{a}{n}, \frac{b}{n} \Rightarrow a < \frac{n}{2}$ & $b > \frac{n}{2}$ $\Rightarrow$ b cannot divide n)

Hence, $f(n) = 3 + 4 + 5 + \ldots.. + (n + 2) = \frac{n(n+5)}{2}$

$$\Rightarrow \quad \frac{f(n+5)}{f(n)} = \frac{(n+5)(n+10)}{n(n+5)} = 1 + \frac{10}{n} \in Z$$

$\Rightarrow$ n = 1, 2, 5, 10

$\Rightarrow$ Required answer = 18

13. Each of the numbers $x_1, x_2, \ldots, x_{101}$ is ± 1. What is the smallest positive value of $\sum\limits_{1 \leq i \leq j \leq 101} x_i x_j$?

Ans. **10**

Sol. Consider the expansion of $(x_1 + x_2 + x_3 + \ldots + x_{101})^2 = x_1^2 + x_2^2 + \ldots + x_{101}^2 + 2 \sum\limits_{1 \leq i < j \leq 101} x_i x_j$

$$\therefore \quad \sum_{1 \leq i < j \leq 101} x_i x_j = \frac{(x_1 + x_2 + \ldots.. + x_{101})^2 - (x_1^2 + x_2^2 + \ldots.. + x_{101}^2)}{2}$$

$\because$ each of x_i is ± 1

$$\therefore \sum_{1 \le i < j \le 101} x_i x_j = \frac{(x_1 + x_2 + \ldots + x_{101})^2 - (101)}{2}$$

Note that to make the difference must be minimum & +ve and also, sum of $x_1 + x_2 + \ldots + x_{101}$ is always odd.

$\therefore$ for minimum, take $x_1 + x_2 + \ldots + x_{101} = 11$.

$$\therefore \sum_{1 \le i < j \le 101} x_i x_j = \frac{11^2 - 101}{2} = 10$$

14. Find the smallest positive integer $n \ge 10$ such that $n + 6$ is a prime and $9n + 7$ is a perfect square.

Ans. **53**

Sol. Let $9n + 7 = K^2$

$$n = \frac{K^2 - 7}{9}$$

$$\text{So, } n + 6 = \frac{K^2 - 7}{9} + 6$$

$$= \frac{K^2 + 47}{9}$$

$$= \frac{K^2 + 2}{9} + 5 \text{ is prime}$$

Now, any perfect square when divided by 9 give remainder 0, 1, 4, 7

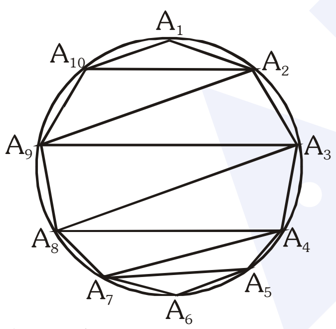
So, K = 22 satisfies the condition.

$\therefore$ n = 53.

15. In how many ways can a pair of parallel diagonals of a regular polygon of 10 sides be selected ?

Ans. **45**

Sol.



Case-1 :

In this case, for vertices A_1, take lines A_2A_{10}, A_3A_9, A_4A_8 & A_5A_7.

All these 4 pair of lines are parallel to each other. i.e. there are ${}^4C_2 = 6$ ways to select a pair of parallel lines.

But when we will see for vertices A_6, Again we are taking same pair of lines.

$$\text{So, Total no. of ways in this case} = \frac{{}^4C_2 \times 10}{2} = 30$$

Case-2 :

In thise case, consider, A_2A_9, A_3A_8, A_4A_7 all these 3 diagonals are parallel to each other.

$$\text{Hence total no. ways of selecting one pair of parallel line is } \frac{{}^3C_2 \times 10}{2} = 15 \text{ ways}$$

$\therefore$ Total number of ways = 30 + 15 = 45 ways.

16. A pen costs Rs. 13 and a note book costs Rs. 17. A school spends exactly Rs. 10000 in the year 2017-18 to buy x pens and y note books such that x and y are as close as possible (i.e., $|x - y|$ is minimum). Next year, in 2018-19, the school spends a little more than Rs. 10000 and buys y pens and x note books. How much more did the school pay ?

Ans. 40

Sol. In 2017 – 18

Cost of pen = 13

Cost of note book = 17

So, $13x + 17y = 10000$ $(x, y \in I^+)$

let us find it's one solution, by trial method.

$[13(3) + 17(-2) = 5] \times 2000$

$\Rightarrow 13(6000) + 17(-4000) = 10000$

So one solution can be

$x = 6000,\ y = -4000$

Both x and y has to be positive integer,

So general solution for x and y will be

$x = 6000 + 17t,\ y = -4000 - 13t$ (t $\in$ whole numbers)

Now, $6000 + 17t > 0$

$$t > \frac{-6000}{17} = -352.94$$

and $-4000 - 13t > 0$

$13t < -4000$

$$t < \frac{-4000}{13} = -307.69$$

$\therefore\ -352.94 < t < -307.69$

$\therefore\ t = (-308, -309, \ldots\ldots, -352)$

By checking certain values and observing the pattern, we find that at $t = -333$, the values of x and y are as close as possible.

So, $t = -333$

$x = 6000 + 17(-333) = 339$

$y = -4000 - 13(-333) = 329$

Hence $x = 339,\ y = 329$

Now, in 2018-19

y pens and x note books.

$\therefore \Rightarrow 17x + 13y$

$\Rightarrow 17(339) + 13(329)$

$\Rightarrow 5763 + 4277$

$\Rightarrow 10,040$

$\therefore$ School pay 40 more this year.

17. Find the number of ordered triples (a,b,c) of positive integers such that $30a + 50b + 70c \le 343$.

Ans. 30

Sol. $30a + 50b + 70c \le 343$

$a, b, c \in I^+$

c can be maximum 4.

When c = 1

$30a + 50b \le 273$

here b can be maximum 5.

So, for b = 1, 30a < 223

$a < \frac{223}{30} \Rightarrow a = 1,2,3,4,5,6,7$

for b = 2, 30a < 173

$a < \frac{223}{30} \Rightarrow a = 1,2,3,4,5$

for b = 3, $30a < 123 \Rightarrow a = 1,2,3,4$

for b = 4, $30a < 73 \Rightarrow a = 1,2$

for b = 5, $30a < 23 \Rightarrow$ No values of a.

Total 18 pairs of (a, b) when c = 1

When c = 2

$30a + 50b \le 203$

here b can be maximum 4.

So, for b = 1, $a \le \frac{152}{30}$, a = 1,2,3,4,5

for b = 2, $a \le \frac{103}{30}$, a = 1,2,3

for b = 3, $a \le \frac{53}{30}$, a = 1

for b = 4, $a \le \frac{3}{30}$, No values of a.

Total 9 pairs of (a, b) when c = 2.

When c = 3

$30a + 50b \le 133$

b can be maximum 2.

for b = 1, $a \le \frac{83}{30}$, a = 1,2

for b = 2, $a \le \frac{33}{30}$, a = 1

Total 3 pairs of (a, b) for c = 3

When c = 4

$30a + 50b \le 63$

b can be maximum 1.

for b = 1, $30a \le 13$

$a \le \frac{13}{30}$, No values of a.

Hence total 18 + 9 + 3 = 30 triples of (a,b,c) are possible.

18. How many ordered pairs (a, b) of positive integers with a < b and $100 \le a, b \le 1000$ satisfy gcd(a, b) : ℓcm (a, b) = 1 : 495 ?

Ans. **20**

Sol. Let a = dp   d = gcd (a,b)

b = dq   and p, q are co-prime

$$\frac{\text{lcm}(a,b)}{\text{gcd}(a,b)} = \frac{dpq}{d} = pq = 495$$

$p.q = 5 \times 9 \times 11$

p < q		possibility
5	99	0 (for d = 20, a = 100 but b = 1980 > 1000, so no solution)
9	55	7 (d = 12 to d = 18)
11	45	13 (d = 10 to d = 22)

$\therefore$ Total possibilities are 20.

19. Let AB be a diameter of a circle and let C be a point on the segment AB such that AC : CB = 6 : 7. Let D be a point on the circle such that DC is perpendicular to AB. Let DE be the diameter through D. If [XYZ] denotes the area of the triangle XYZ, find [ABD]/[CDE] to the nearest integer.

Ans. **13**

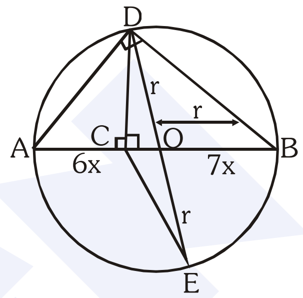
Sol. $\frac{AC}{BC} = \frac{6}{7}$

OC = 7x − r

OC = r − 6x

7x − r = r − 6x

$\frac{13x}{2} = r$



OC is the median of Δ CDE

$\therefore$ ar (ΔCDE) = 2ar(ΔDCO)

$ar(\Delta DCO) = \frac{1}{2} ar(\Delta CDE)$

$\Rightarrow \frac{1}{2} \times CD \times OC = \frac{1}{2} ar(\Delta CDE)$

$\Rightarrow$ ar(ΔCDE) = CD × OC

OD is the median of ΔABD

$\therefore$ ar(ΔABD) = 2ar(ΔBOD) =

$$= 2 \times \frac{1}{2} \times r \times CD$$

Now, $\frac{[ABD]}{[CDE]} = \frac{r \times CD}{CD \times OC} = \frac{r}{OC} = \frac{r}{r-6x} = \frac{r}{r - 6 \times \frac{2r}{13}} = \frac{r}{\frac{13r-12r}{13}} = 13$.

20. Consider the set E of all natural numbers n such that when divided by 11, 12, 13, respectively, the remainders, in that order, are distinct prime numbers in an arithmetic progression. If N is the largest number in E, find the sum of digits of N.

Ans. **Bonus**

Sol. Possible primes are 3, 5, 7

$x \equiv 3 \pmod{11}$

$x \equiv 5 \pmod{12}$

$x = 7 \pmod{13}$

By chinese remainder theorem we have,

$x \equiv a_1m_1y_1 + a_2m_2y_2 + a_3m_3y_3 \pmod{11 \times 12 \times 13}$

here $\begin{array}{l|l|l} a_1 = 3 & m_1 = 156 & 156y_1 \equiv 1 \pmod{11} \\ a_2 = 5 & m_2 = 143 & 2y_1 \equiv 12 \pmod{11} \\ a_3 = 7 & m_3 = 132 & \therefore y_1 = 6 \end{array}$

$\begin{array}{r|l} 143y_2 \equiv 1 \pmod{12} & 132y_3 \equiv 1 \pmod{13} \\ -y_2 \equiv 1 \pmod{12} & 2y_3 \equiv 14 \pmod{13} \\ \therefore y_2 = 11 & \therefore y_3 = 7 \end{array}$

$\therefore$ $x \equiv 3 \times 156 \times 6 + 5 \times 143 \times 11 + 7 \times 132 \times 7 \pmod{11 \times 12 \times 13}$

$x \equiv 17141 \pmod{1716}$

$x \equiv 1697 + 1716k \pmod{1716}$, $\forall k \in I$

Hence there does not exists largest such number.

Same can be concluded for the case of 3, 7, 11.

21. Consider the set E = {5,6,7,8,9}. For any partition {A, B} of E, with both A and B non-empty, consider the number obtained by adding the product of elements of A to the product of elements of B. Let N be the largest prime number among these numbers. Find the sum of the digits of N.

Ans. 17

Sol. According to given condition 6, 8 and 9 should be in a single set.

3 numbers are possible.

$7 \times 5 + 6 \times 8 \times 9 = 35 + 432 = 467$

$7 + 432 \times 5 = 7 + 2160 = 2167$ divisible by 11

$5 + 432 \times 7 = 5 + 3024 = 2029$ divisible by 13

467

So, required answer is $4 + 6 + 7 = 17$

22. What is the greatest integer not exceeding the sum $\sum_{n=1}^{1599} \frac{1}{\sqrt{n}}$?

Ans. 78

Sol. Let $S = \sum_{r=1}^{n} \frac{1}{\sqrt{r}}$; $n = 1599$

Now, $\frac{1}{\sqrt{r}} = \frac{2}{2\sqrt{r}} = \frac{2}{\sqrt{r} + \sqrt{r}} < \frac{2}{\sqrt{r} + \sqrt{r-1}}$

$= 2\left(\sqrt{r} - \sqrt{r-1}\right)$

So, $\frac{1}{\sqrt{1}} + t_2 + t_3 + ... + t_n < 1 + 2(\sqrt{n} - 1)$

$= 2\sqrt{1599} - 1 < 79$

So, $S < 79$

Also, $\frac{1}{\sqrt{r}} = \frac{2}{\sqrt{r} + \sqrt{r}} > \frac{2}{\sqrt{r} + \sqrt{r+1}} = 2\left(\sqrt{r+1} - \sqrt{r}\right)$

So, $t_1 + t_2 + ... + t_n > 2\left(\sqrt{n+1} - 1\right) = 78$

Thus, $78 < S < 79$

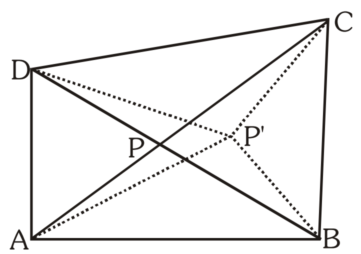
$\Rightarrow [S] = 78$

23. Let ABCD be a convex cyclic quadrilateral. Suppose P is a point in the plane of the quadrilateral such that sum of its distances from the vertices of ABCD is the least. If

{PA, PB, PC, PD} = {3,4,6,8}, what is the maximum possible area of ABCD ?

Ans. 55

Sol.



We claim that P should be point of intersection diagonals of AC and BD.

Proof : P'A + P'B + P'C + P'D

= (P'A + P'C) + (P'B + P'D)

$\geq$ AC + BD

Hence, equality holds when P' is collinear with A and C as well as with B and D i.e. P' = P.

Let PA = a, PB = b, PC = c and PD = d

$\{a,b,c,d\} = \{3,4,6,8\}$

Now, say APD = θ

So, $[ABCD] = \frac{1}{2}\sin\theta\,(ab + bc + cd + da)$

$= \frac{1}{2} \times 1\,(8\times 6 + 8\times 4 + 6\times 3 + 4\times 3)$

= 55

24. A $1\times n$ rectangle ($n \geq 1$) is divided into n unit (1×1) squares. Each square of this rectangle is coloured red, blue or green. Let f(n) be the number of colourings of the rectangle in which there are an even number of red squares. What is the largest prime factor of f(9)/f(3)? (The number of red squares can be zero.)

Ans. 37

Sol. $f(9) = {}^9C_0 \times 2^9 + {}^9C_2 \times 2^7 + {}^9C_4 \times 2^5 + {}^9C_6 \times 2^3 + {}^9C_8 \times 2$

$= \frac{(2+1)^9 + (2-1)^9}{2} = \frac{3^9+1}{2}$

Also $f(3) = {}^3C_0 \times 2^3 + {}^3C_2 \times 2^1 = 14$

Now, $\frac{f(9)}{f(3)} = \frac{3^9+1}{28} = \frac{(3^3+1^3)(3^6-3^3+1)}{28} = 703 = 37\times 19$

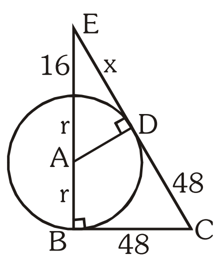
Hence, required answer = 37

25. A village has a circular wall around it, and the wall has four gates pointing north, south, east and west. A tree stands outside the village, 16 m north of the north gate, and it can be just seen appearing on the horizon from a point 48 m east of the south gate. What is the diameter, in meters, of the wall that surrounds the village?

Ans. 48

Sol. $\Delta EBC \sim \Delta EDA$

So, $\frac{16+r}{48+x} = \frac{r}{48} = \frac{x}{16+2r}$



Eliminating x :

$r(16 + 2r) = 48 \times \frac{16+48}{r}$

$\Rightarrow r^2(8 + r) = 24^2 \times (8 + 24)$

$\Rightarrow r = 24$

$\Rightarrow$ Diameter = 48

26. Positive integers x, y, z satisfy xy + z = 160. Compute the smallest possible value of x + yz.

Ans. **50**

Sol. Given that xy + z = 160

We need to find the smallest possible value of x + yz

So we need to make y, z as minimum as possible.

First solution :

By hit and trial method, keeping x, y, z are integers.

Z = 1, not possible

$Z = 2 \Rightarrow xy = 158 \Rightarrow x = 79, y = 2$

$Z = 3 \Rightarrow$ Not possible

$Z = 4 \Rightarrow xy = 156 \Rightarrow x = 26, y = 6$ is giving a value of

x + yz = 50 as minimum.

for all other integer z, we will get 'x + yz' more than 50.

So smallest possible value of x + yz = 50

Second method :

Put xy = a

then z = 160 – a

$$\text{Now } x + yz = \frac{a}{y} + y(160 - a)$$

Now for smallest value using

AM $\geq$ GM we have

$$\Rightarrow \frac{a}{y} + y(160 - a) \geq 2\sqrt{\frac{a}{y}[y(160-a)]}$$

$$\Rightarrow x + yz \geq 2\sqrt{a(160-a)}$$

Now to maximize 'a' value we will check values of 'a' from heighest values. So putting a = 156, gives x + yz $\geq 49.96 \cong 50$.

$\therefore$ minimum value of x + yz = 50

27. We will say that a rearrangement of the letters of a word has no fixed letters if, when the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, HBRATA is a rearrangement with no fixed letters of BHARAT. How many distinguishable rearrangements with no fixed letters does BHARAT have ? (The two As are considered identical)

Ans. **84**

Sol.

$$\frac{A}{B}\ \frac{A}{H}\ \frac{\ }{A}\ \frac{\ }{R}\ \frac{\ }{A}\ \frac{\ }{T}$$

If we place 2 A's in any two places other than their original one (let's say B and H without loss of generality) number of ways to arrange R, T, B and H among four places.

3×2 + $2 \times 2 \times 2$

(for R or T) (for any of two A's)

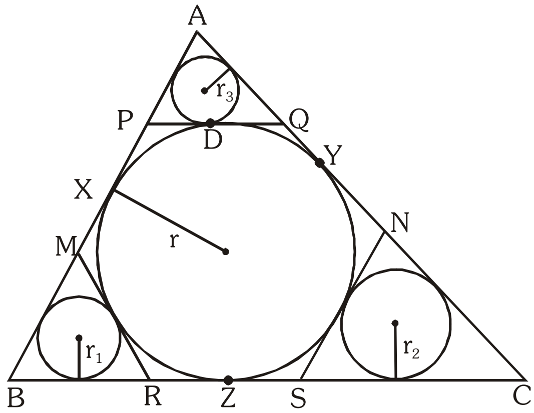
6 + 8 = 14

So total ways ${}^4C_2 \times 14 = 84$

28. Let ABC be a triangle with sides 51, 52, 53. Let Ω denote the incircle of ΔABC. Draw tangents to Ω which are parallel to the sides of ABC. Let r_1, r_2, r_3 be the inradii of the three corner triangles so formed. Find the largest integer that does not exceed $r_1 + r_2 + r_3$.

Ans. **15**

Sol.



Here, $r(\Delta ABC) = \dfrac{ar(\Delta ABC)}{S(ABC)}$ where S = semi perimeter

$ar(\Delta ABC) = \sqrt{78 \times 27 \times 26 \times 25} = 1170$

and $S(ABC) = 78$

$\therefore \quad r(\Delta ABC) = \dfrac{1170}{78} = 15$

Note that tangents to the smaller circle are parallel to the sides of a triangle ABC.

$\therefore$ Each of 3 smaller triangles are similar to ΔABC.

$$\therefore \quad \frac{PQ}{BC} = \frac{r_3}{r} \quad ....(i)$$

$$\frac{RM}{AC} = \frac{r_1}{r} \quad ....(ii)$$

$$\frac{SN}{AB} = \frac{r_2}{r} \quad ....(iii)$$

Adding these, we get

$$(r_1 + r_2 + r_3)\left(\frac{1}{r}\right) = \frac{PQ}{BC} + \frac{RM}{AC} + \frac{SN}{AB} \quad ....(iv)$$

Claim : $\dfrac{PQ}{BC} + \dfrac{RM}{AC} + \dfrac{SN}{AB} = 1$

Proof : We know,

$AX = AY = S - a$

Also, $AX = AP + PX = AP + PD$

and $AY = AQ + AY = AQ + QD$

$\Rightarrow$ Perimeter of $\Delta APQ = 2AX$

Now, $\dfrac{PQ}{BC} = \dfrac{\text{Perimeter}(\Delta APQ)}{\text{Perimeter}(\Delta ABC)} = \dfrac{2(S-a)}{2S} \quad ....(v)$

Similarly $\dfrac{MR}{AC} = \dfrac{2(S-b)}{2S} \quad ....(vi)$

& $\frac{SN}{AB} = \frac{2(S-C)}{2S}$ ....(vii)

Adding there 3 equations. we get

Now, $\frac{PQ}{BC} + \frac{RM}{AC} + \frac{SN}{AB} = \frac{3S-(a+b+c)}{3} = 1$

From eq^n (iv), $r_1 + r_2 + r_3 \left(\frac{1}{r}\right) = 1$

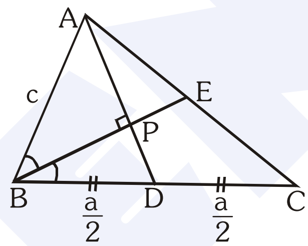
$\Rightarrow r_1 + r_2 + r_3 = r$

$\Rightarrow r_1 + r_2 + r_3 = 15$

29. In a triangle ABC, the median AD (with D on BC) and the angle bisector BE (with E on AC) are perpendicular to each other. If AD = 7 and BE = 9, find the integer nearest to the area of triangle ABC.

Ans. 47

Sol. $\Delta APB \cong \Delta DPB$



$\Rightarrow$ AP = PD = $\frac{7}{2}$

& $c = \frac{a}{2}$

Now BE = 9

$\Rightarrow \frac{2ac}{a+c}\cos\frac{B}{2} = 9$ ...(i)

Also, In ΔABP : $c\sin\frac{B}{2} = AP$

$\Rightarrow c\sin\frac{B}{2} = \frac{7}{2}$ ...(ii)

(i) $\times$ (ii) :

$\left(2ac\sin\frac{B}{2}\cos\frac{B}{2}\right)\frac{c}{a+c} = \frac{63}{2}$

$\Rightarrow \frac{1}{2}ac\sin B = \frac{189}{4}$

$\Rightarrow \Delta = 47.25$

So, required answer is 47.

30. Let E denote the set of all natural numbers n such that $3 < n < 100$ and the set $\{1,2,3,.....,n\}$ can be partitioned in to 3 subsets with equal sums. Find the number of elements of E.

Ans. 64

Sol. For $\frac{n(n+1)}{2}$ to be partitioned in 3 sets with equal sum.

Either of n or n + 1 should be divisible by 3

i.e. n = 3k or 3k – 1

So n = 3k = 6, 9, 12, ... 99

$\Rightarrow$ 3k – 1 = 5, 8, 11, ... 98

Total term = 64