SOLUTION
THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
KAPREKAR CONTEST - FINAL - SUB JUNIOR
CLASS - VII & VIII

Instructions:
1. Answer all the questions.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer paper itself.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

1. Let a_n be the units place of $1^2 + 2^2 + 3^2 + \dots\dots\dots + n^2$. Prove that the decimal $0.a_1a_2a_3.....a_n......$ is a rational number and represent it as $\frac{p}{q}$, where p and q are natural numbers.

Sol. $a_n = 1^2 + 2^2 + 3^2 + \dots + n^2$ (units digit)

$a_1 = 1, a_2 = 5, a_3 = 4, a_4 = 0, a_5 = 5, a_6 = 1, a_7 = 0, a_8 = 4, a_9 = 5, a_{10} = 5, a_{11} = 6, a_{12} = 0, a_{13} = 9, a_{14} = 5, a_{15} = 0, a_{16} = 6, a_{17} = 5, a_{18} = 9, a_{19} = 0, a_{20} = 0, a_{21} = 1, a_{22} = 5, a_{23} = 4, a_{24} = 0$

Similarly all unit digits are repeating same as a_1 to a_{20}.

$$\frac{p}{q} = 0.\overline{15405104556095065900}$$

$$\frac{p}{q} = \frac{15405104556095065900}{99999999999999999999}$$

p, q are natural numbers.

2. (a) Find the positive integers m, n such that $\frac{1}{m} + \frac{1}{n} = \frac{3}{17}$

(b) Find the positive integers m, n, p such that $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = \frac{3}{17}$.

(c) Using this idea, prove that we can find for any positive integer k, k distinct integers, $n_1, n_2, ...n_k$ such that $\frac{1}{n_1} + \frac{1}{n_2} + ...\frac{1}{n_k} = \frac{3}{17}$.

Sol. (a) $\frac{1}{m} + \frac{1}{n} = \frac{3}{17}$

$$\frac{17}{3m} + \frac{17}{3n} = 1$$

$(3m - 17)(3n - 17) = 289 = 1 \times 289$

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$3m - 17 = 1, \qquad 3n - 17 = 289$

$3m = 18, \qquad 3n = 306$

$m = 6, \qquad n = 102$

$3m - 17 \neq 17 \qquad 3n - 17 \neq 17, \qquad m, n \in z$

$\therefore (m, n) = (6, 102) = (102, 6)$

(b) $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = \frac{3}{17}$

$m \leq n \leq p$

$\frac{1}{m} \geq \frac{1}{n} \geq \frac{1}{p}$

$\frac{3}{m} \geq \frac{3}{17} \Rightarrow \frac{1}{m} \geq \frac{1}{17} \Rightarrow m \leq 17$

$\frac{1}{m} \leq \frac{3}{17} \Rightarrow \frac{17}{3} \leq m$

$\therefore 6 \leq m \leq 17$

for $m = 6$, $\frac{1}{6} + \frac{1}{n} + \frac{1}{p} = \frac{3}{17}$

$\frac{1}{n} + \frac{1}{p} = \frac{1}{102}$

$(n - 102)(p - 102) = 102 \times 102$

$\Rightarrow m = 6, n = 204, p = 204$

Similarly for $m = 7, 8, \ldots 17$ equations could be solved.

(c) $n_1 \leq n_2 \leq n_3 \leq \ldots \leq n_k$

$\frac{1}{n_1} \geq \frac{1}{n_2} \geq \frac{1}{n_3} \geq \ldots . \frac{1}{n_k}$

$\frac{k}{n_1} \geq \frac{3}{17} \Rightarrow n_1 \leq \frac{17}{3} k$

$\frac{1}{n_1} \leq \frac{3}{17} \Rightarrow \frac{17}{3} \leq n_1$

$\Rightarrow \frac{17}{3} \leq n_1 \leq \frac{17k}{3}$

Solving equations we can find for any positive integer k, k distinct integers.

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3. Does there exist a positive interger which is a multiple of 2019 and whose sum of the digits is 2019? If no, prove it. If yes, give one such number.

Sol. Multiples of 2019 are 2019, 4038, 6057, 8076, 10095, 12114 .... So on and sum of their digits are 12, 15, 18, 21, 16, 9 respectively.

Now consider a number 807612114 whose some of digits is 30.

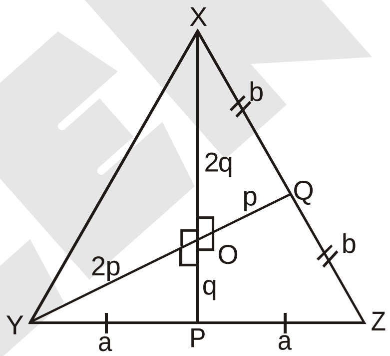
Now take '807612114' 67 times and '12114' one more time such that

$\underbrace{807612114 \ldots\ldots\ldots 12114}_{67 \text{ times}}12114 \rightarrow$ Sum of digit of this number is 2019 and number is also divisible by 2019.

4. In a triangle XYZ, the medians drawn through X and Y are perpendicular. Then show that XY is the smallest side of XYZ.

Sol. Let YZ and XZ are 2a and 2b respectively, XP and YQ are 3q and 3p respectively.

In ΔXOY ; $XY = \sqrt{4p^2 + 4q^2}$ ....(1)



In ΔYOP ; $a = \sqrt{4p^2 + q^2}$

$YZ = 2a = 2\sqrt{4p^2 + q^2}$

$= \sqrt{16p^2 + 4q^2}$

$= \sqrt{4p^2 + 4q^2 + 12p^2}$ ......(2)

In ΔXOQ ; $b = \sqrt{p^2 + 4q^2}$

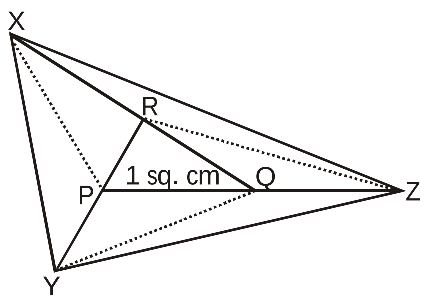
$XZ = 2b = 2\sqrt{p^2 + 4q^2}$

$= \sqrt{4p^2 + 16q^2}$

$= \sqrt{4p^2 + 4q^2 + 12q^2}$ .....(3)

From (1), (2), & (3) we can say XY is the smallest side.

5. Let Δ PQR be a triangle of area 1 cm². Extend QR to X such that QR = RX; RP to Y such that RP = PY and PQ to Z such that PQ = QZ. Find the area of ΔXYZ.



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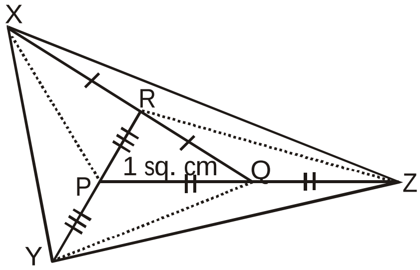
Sol.



In $\Delta PQX \rightarrow PR$ is median

$\therefore$ ar (ΔPXR) = ar (ΔPQR) = 1 cm^2 ......(1)

In Δ YRX $\rightarrow$ XP is median

$\therefore$ ar (ΔYPX) = ar (ΔPXR) = 1 cm^2 ......(2)

In Δ YQR $\rightarrow$ PQ is median

$\therefore$ ar $(\Delta$ YPQ$)$ = ar $(\Delta$ PQR$)$ = 1 cm^2 ....(3)

In Δ YPZ $\rightarrow$ YQ is median

$\therefore$ ar $(\Delta$ YPQ$)$ = ar $(\Delta$ YQZ$)$ = 1 cm^2 ....(4)

In Δ PRZ $\rightarrow$ RQ is median

$\therefore$ ar $(\Delta$ PQR$)$ = ar $(\Delta$ QRZ$)$ = 1 cm^2 ...(5)

In Δ XQZ $\rightarrow$ RZ is median

ar $(\Delta$ RQZ$)$ = ar (ΔXRZ) = 1 cm^2 ...(6)

Using eq (1), (2), (3), (4), (5), (6)

area of ΔXYZ = 7 cm^2

6. Find the real numbers x and y given that $x - y = \frac{3}{2}$ and $x^4 + y^4 = \frac{2657}{16}$

Sol. $x - y = \frac{3}{2}$, $x^4 + y^4 = \frac{2657}{16}$

$\Rightarrow x^2 + y^2 - 2xy = \frac{9}{4}$

$\Rightarrow x^2 + y^2 = \frac{9}{4} + 2xy$

$\Rightarrow x^4 + y^4 + 2x^2y^2 = \frac{81}{16} + 4x^2y^2 + 9xy$

$\Rightarrow \frac{2657}{16} - \frac{81}{16} = 2x^2y^2 + 9xy$

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$\Rightarrow \dfrac{2576}{16} = 2x^2y^2 + 9xy$

Let $xy = t$

$2t^2 + 9t - 161 = 0$

$t = \dfrac{-23}{2}, 7$

Case-I

$xy = 7, x - y = \dfrac{3}{2}$

$\dfrac{7}{y} - y = \dfrac{3}{2}$

$2y^2 + 3y - 14 = 0$

$(y - 2)(2y + 7) = 0$

$y = 2, -7/2$

For $y = 2, x = \dfrac{7}{2}$

$y = \dfrac{-7}{2}, x = -2$

Case-II

$xy = \dfrac{-23}{2}, x - y = \dfrac{3}{2}$

$\dfrac{-23}{2y} - y = \dfrac{3}{2}$

$2y^2 + 3y + 23 = 0$

No real value of y exists.

$\therefore$ Possible real values are $x = -2, y = -\dfrac{7}{2}$ and $x = \dfrac{7}{2}, y = 2$

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7. The difference of the eight digit number ABCDEFGH and the eight digit number GHEFCDAB is divisible by 481. Prove that C = E and D = F.

Sol. Acc. to question [ABCDEFGH – GHEFCDAB]

$= (10^7A+10^6B + 10^5C + 10^4D + 10^3E + 10^2F + 10^1 G + H)$

$-(10^7G + 10^6H + 10^5E + 10^4F + 10^3C + 10^2D + 10^1 A + B)$

$= 10 (10^6 - 1) A + (10^6 - 1) B + 10^3 (10^2 - 1) C + 10^2 (10^2 - 1) D$

$+ 10^3(1 - 10^2) E + 10^2 (1 - 10^2) F + 10 (1 - 10^6) G + (1 - 10^6) H.$

$= (10^6 - 1) [10A + B - 10G - H] + (10^2 - 1) (10^3C + 10^2D - 10^3 E - 10^2F)$

$$= \underbrace{999999\,(10A + B - 10G - H)}_{\substack{\text{divisible by 481}\\ \text{as } 999999 = 481 \times 2079}} + \underbrace{99\times10^3\,(C-E) + 99\times10^2\,(D-F)}_{\text{Let }(k)}$$

$\Rightarrow k = 99000 (C-E) + 9900 (D-F) = 9900 (10C - 10E + D-F)$

Here k should be multiple of 481.

Now, $k = 9900 (10C - 10E + D - F)$

$$\underbrace{20\times481(10C-10E+D-F)}_{\text{Divisible by 481}} + \underbrace{280[10C-10E+D-F]}_{m}$$

(481 = 37 × 13)

37, 13 are not factors of 280 and maximum possible value of 10C – 10E + D – F is 99, which can not be a multiple of 37 and 13 simultaneously.

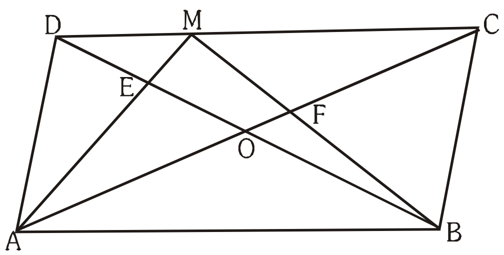
$\therefore\ 10C - 10E + D - F = 0$

$\Rightarrow\ 10 (C - E) = (F - D)$

This is only possible when $C - E = 0$ and $F - D = 0$

$\therefore C = E$ and $F = D$

8. ABCD is a parallelogram with area 36 cm^2. O is the intersection point of the diagonals of the parallelogram. M is a point on DC. The intersection point of AM and BD is E and the intersection point of BM and AC is F. The sum of the areas of the triangles AED and BFC is 12 cm^2. What is the area of the quadrilateral EOFM?



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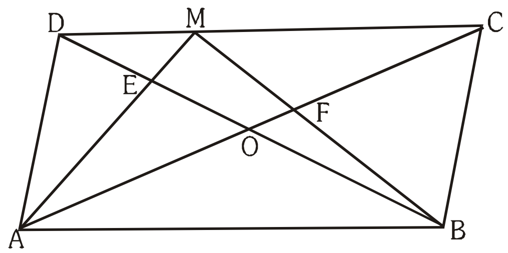
Sol. ar.($||^{gm}$ ABCD) = 36 cm^2

$\therefore$ ar. (ΔABM) = 18 cm^2



[If a triangle and a parallelogram are on the same base and between the same parallels then the area of triangle is half of area of parallelogram.]

ar(ΔAOB) = ar(ΔBOC) = ar(ΔCOD) = ar(ΔAOD) = 9 cm^2

ar(ΔAOB) + ar(ΔBOC) = 18 cm^2 ....(1)

ar(ΔAED) + ar(ΔBFC) = 12cm^2 (given) ....(2)

ar(ΔAOE) + ar(ΔBOF) = [ar(ΔAOD) + ar(ΔBOC)] – [ar(ΔAED) + ar(ΔBFC)]

= 18 cm^2 – 12cm^2

= 6 cm^2

ar (quad. EOFM) = ar(ΔAMB) – [ar(ΔAOB) + ar(ΔAOE) + ar(ΔBOF)]

= 18 – [9 + 6] cm^2

= (18 – 15) cm^2

= 3 cm^2