SOLUTION
THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
BHASKARA CONTEST - FINAL - JUNIOR
CLASS - IX & X

Instructions:

1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

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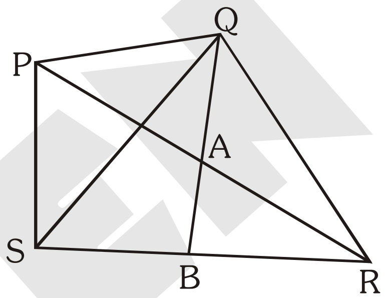
1. In a convex quadrilateral PQRS, the areas of triangles PQS, QRS and PQR are in the ratio 3 : 4 : 1. A line through Q cuts PR at A and RS at B such that PA: PR = RB : RS. Prove that A is the midpoint of PR and B is the midpoint of RS.

Sol. [PQS] = 3x

[QRS] = 4x

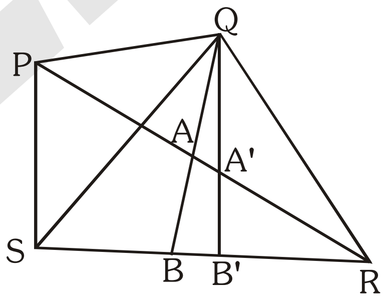
[PQR] = x



Since $\frac{PA}{PR} = \frac{RB}{RS}$

$\frac{[PQA]}{[PQR]} = \frac{[QRB]}{[QRS]} \Rightarrow \frac{[PQA]}{\cancel{x}} = \frac{[QRB]}{4\cancel{x}}$

4[PQA] = [QRB] .....(i)



Since [PQR] = x

So one solution is [PQA] = $\frac{x}{2}$ then

then [QRB] = $4 \times \frac{x}{2} = 2X = \frac{1}{2}$[QRS]

Thus in ΔQPR, [PQA] = $\frac{1}{2}$[PQR] and in ΔQSR, [QRB] = $\frac{1}{2}$[QRS]

Thus A and B are the mid points of PR and RS respectively.

Let if possible there is another line QA'B' where A' is on PR and B' is on RS.

As we more from B to B' either [PQA'] increases and [QRB'] decreases or vice versa so which will not satisfy eq.(i).

2. Given positive real numbers a, b, c, d such that cd = 1. Prove that there exist at least one positive integer m such that ab ≤ m^2 ≤ (a + c) (b + d).

Sol. To prove : ab ≤ m^2 ≤ (a + c) (b + d)

here, (a + c) (b+ d) = ab + ad + bc + 1

$\geq ab + 2\sqrt{ab} + 1$ (by applying AM – GM on ad + bc and using cd = 1)

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$= \left(\sqrt{ab}+1\right)^2$

Now, if $\sqrt{ab}$ = an integer then m = $\sqrt{ab}$ satisfies else, ($\sqrt{ab}$ + 1) = m satisfies.

3. Find the number of permutations $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ of the integers –3, –2, –1, 0, 1, 2, 3, 4 that satisfy the chain of inequalities $x_1x_2 \le x_2x_3 \le x_3x_4 \le x_4x_5 \le x_5x_6 \le x_6x_7 \le x_7x_8$.

Sol. $\{x_1, x_2, x_3 \,.........x_8\} = \{-3, -2, ...3,4\}$

$x_1x_2 \le x_2x_3 \le x_3x_4 \,......... \le x_7x_8$

In any permutations either exactly 3 or 2 or only 1 –ve number can lie some where on the right of 0. But in all these cases required chain of inequalities are not established.

Hence all –ve numbers must be somewhere left of zero.

Now sequence will start from +ve or –ve.

Suppose $x_1 > 0 \Rightarrow x_2 < 0$

Given $x_1x_2 \le x_2x_3 \Rightarrow x_1 \ge x_3$

Hence +ve numbers left of zero must be in descending order.

Similarly –ve numbers left of zero must be ascending order.

Only sequence possible is

Case-1 + – + – + – + 0 → No. of ways = 1

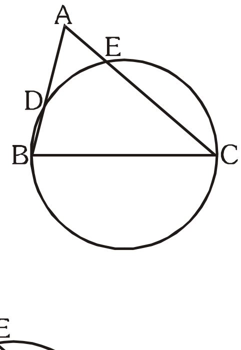
Case-2 + – + – + – 0 + → No. of ways = ${}^4C_1 = 4$

Case-3 – + – + – 0 + + → No. of ways = ${}^4C_2 \times 2 = 12$

Case-4 – + – + – + 0 + → No. of ways = ${}^4C_1 = 4$

Total no. of permutations = 1 + 4 + 12 + 4 = 21

4. In the figure, BC is a diameter of the circle, where BC = $\sqrt{257}$, BD = 1, and DA = 12.



Find the length of EC and hence find the length of the altitude from A to BC.

Sol. $CD^2 = BC^2 - BD^2 = AC^2 - AD^2$

$257 - 1 = (x + y)^2 - (12)^2$

$256 + 144 = (x + y)^2$

$400 = (x + y)^2$

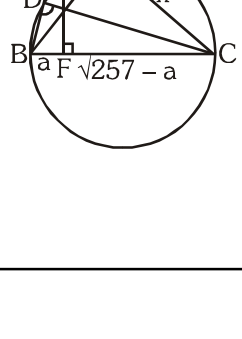
$x + y = 20$ .....(i)

$\Rightarrow BE^2 = AB^2 - AE^2 = BC^2 - CE^2$

$169 - y^2 = 257 - x^2$

$x^2 - y^2 = 257 - 169$

$(x - y)(x + y) = 88$



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$\Rightarrow (x - y) = \frac{88}{20} = \frac{22}{5}$ ... (ii)

Using equation (i) and (ii) we get, $x = \frac{61}{5}$

Now, $CD^2 = BC^2 - BD^2 = 257 - 1 = 256$

$CD = 16$

$\therefore$ Area of $\Delta ABC = \frac{1}{2} BC \times AF = \frac{1}{2} AB \times CD$

$\Rightarrow \sqrt{257}\, AF = 13 \times 16$

$\Rightarrow AF = \frac{208}{\sqrt{257}}$

5. A math contest consists of 9 objective type questions and 6 fill in the blanks questions. From a school some number of students took the test and it was noticed that all students had attempted exactly 14 out of the 15 questions. Let $O_1, O_2, \ldots, O_9$ be the nine objective questions and $F_1, F_2, \ldots, F_6$ be the six fill in the blanks questions. Let a_{ij} be the number of students who attempted both questions O_i, and F_j, If the sum of all the a_{ij}, $i = 1, \ldots, 9$ and $j = 1, \ldots, 6$ is 972, then find the number of students who took the test in the school.

Sol. A student has attempted 14 question it means 1 question is left by each student. Let number of students who leave 1 objective type question = x and number of students who leave 1 fill up = y

According to the counting

Number of times a student leaving 1 objective question gets counted = $8 \times 6 = 48$

Number of students leaving 1 fill up question gets counted = $9 \times 5 = 45$

$\therefore$ ATQ

$48x + 45y = 972$

$16x + 15y = 324$

$3|x$ and $4|y$

let $x = 3a$, $y = 4b$

$\therefore 4a + 5b = 27$

$\Rightarrow a = 3, b = 3$

$n = 3a + 4b$

$= 9 + 12$

$= 21$

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6. Find all positive integer triples (x, y, z) that satisfy the equation $x^4 + y^4 + z^4 = 2x^2y^2 + 2y^2z^2 + 2z^2x^2 - 63$

Sol. $x, y, z \in N$

$x^4 + y^4 + z^4 = 2x^2y^2 + 2y^2z^2 + 2z^2x^2 - 63$

$\Rightarrow x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 = -63$

$\Rightarrow (x^2 - y^2 - z^2)^2 - (2yz)^2 = -63$

$\Rightarrow (x^2 - y^2 - z^2 - 2yz)(x^2 - y^2 - z^2 + 2yz) = -63$

$\Rightarrow (x^2 - (y+z)^2)(x^2 - (y-z)^2) = -63$

$\Rightarrow -(x + y + z)(-x + y + z)(x + y - z)(x - y + z) = -63$

$\Rightarrow (x + y + z)(-x + y + z)(x + y - z)(x - y + z) = 63$

	A x+y+z	B x+y-z	C y+z-x	D z+x-y	
I	63	1	1	1	
II	63	–1	–1	1	3 cases
III	9	7	1	1	3 cases
IV	9	7	–1	–1	3 cases
V	21	3	1	1	3 cases
VI	21	3	–1	–1	3 cases
VII	21	–3	–1	1	3 cases
VIII	7	3	3	1	3 cases
IX	7	–3	–3	1	3 cases
X	7	3	–3	–1	3 cases

For case 1. B + C + D

$x + y + z = 3$ but $x + y + z = 63$. Hence there is no solution.

For case 2. B + C + D

$x + y + z = -1$ but $x + y + z = 63$. Hence there is no solution.

Similarly Case 4, 5, 6, 7, 9, 10 is also inconsistent.

From case 3 B + C +D

$x + y + z = 9$, also from $A = x + y + z = 9$

$x + y - z = 7$

$y + z - x = 1$

$z + x - y = 1$

on solving

$x = 4, y = 4, z = 1$

$(x, y, z) \approx (4, 4, 1)$

From cyclicity,

$(x, y, z) : (4, 4, 1), (4, 1, 4) \& (1, 4, 4)$

From case 8 B + C +D

$x + y + z = 7$, also from $A = x + y + z = 7$

$x + y - z = 3$

$y + z - x = 3$

$z + x - y = 1$

on solving

$x = 2, y = 3, z = 2$

$(x, y, z) \approx (2, 3, 2)$

From cyclicity,

$(x, y, z) : (2, 3, 2), (2, 2, 3) \& (3, 2, 2)$

Alternate solution

Let us form a quadratic in x^2

$x^4 - 2(y^2 + z^2)x^2 + (y^4 + z^4 - 2y^2z^2 + 63) = 0$

Since x is an integer, its discriminant must be a perfect square.

$\therefore D = (2(y^2 + z^2))^2 - 4(1)(y^4 + z^4 - 2y^2z^2 - 63)$

$= 4[y^4 + z^4 + 2y^2z^2] - 4y^4 - 4z^4 + 8y^2z^2 - 252$

$= 4y^4 + 4z^4 + 8y^2z^2 - 4y^4 - 4z^4 + 8y^2z^2 - 252$

$= 16y^2z^2 - 252$

Let it equals to k^2

$\therefore (4yz)^2 - 252 = (k)^2$

$(4yz)^2 - (k)^2 = 252$

$(4yz + k)(4yz - k) = 252$

Sum of factors in LHS is even, so

Case 1 : $(4yz + k)(4yz - k) = 126 \times 2$

$4yz + k = 126$

$4yz - k = 2$

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Which gives k = 62 and yz = 16
yz = 16 implies y = 4, z = 4
y = 16, z = 1
y = 8, z = 2

Now, $x^2 = \dfrac{2(y^2+z^2)\pm 62}{2}$ (By Shreedharacharya formula)

Checking for various values of y, z such that x^2 and hence x comes as positive integer, we get integeral x at y = z = 4, as $x^2 = \dfrac{2(16+16)-62}{2} = 1.$

So the solution is (1, 4, 4) as well as it's permutations (4, 1, 4) ; (4, 4, 1).
Case 2 : (4yz + k) (4yz – k) = 18 × 14
4yz + k = 18
4yz – k = 14
Which gives k = 2 and yz = 4
yz = 4 implies y = 4, z = 1
y = 2, z = 2

Now, $x^2 = \dfrac{2(y^2+z^2)\pm 2}{2}$ (By Shreedharacharya formula)

Checking for various values of y, z such that x^2 and hence x comes as positive integer, we get integeral x at y = z = 2, as $x^2 = \dfrac{2(4+4)+2}{2} = 9. \Rightarrow x = 3$

So the solution is (3, 2, 2) as well as it's permutations (2, 3, 2) ; (2, 2, 3).
Similarly from case 3 i.e. (4yz + k) (4yz – k) = 42 × 6, we will get the above solutions.

7. The perimeter of ΔABC is 2 and its sides are BC = a, CA = b, AB = c.

Prove that $abc + \dfrac{1}{27} \geq ab + bc + ca - 1 \geq abc.$

Sol. Given, a + b + c = 2

To prove : $abc + \dfrac{1}{27} \geq ab + bc + ca - 1 \geq abc$

a, b, c are sides of a triangle.

$\therefore\ a < b + c$
$a < 2 - a$
$2a < 2$
$a < 1$

So, $1 - a > 0$

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Similarly $1 - b > 0$, $1 - c > 0$

$\therefore \quad (1 - a)(1 - b)(1 - c) \geq 0$

$1 - (a + b + c) + (ab + bc + ca) - (abc) \geq 0$

$1 - 2 + ab + bc + ca - abc \geq 0$

$ab + bc + ca - 1 \geq abc$

This completes the second part.

Also for first part, we have to prove

$$\frac{1}{27} \geq ab + bc + ca - 1 - abc$$

which is eqivalent to

$$\frac{1}{27} \geq (1 - a)(1 - b)(1 - c)$$

Which can be proved using AM – GM inequality, as

$1 - a + 1 - b + 1 - c \geq 3[(1 - a)(1 - b)(1 - c)]^{1/3}$

$$\frac{1}{3} \geq [(1 - a)(1 - b)(1 - c)]^{1/3}$$

Cubing both sides, we get

$$\frac{1}{27} \geq (1 - a)(1 - b)(1 - c)$$

Hence proved

8. A circular disc is divided into 12 equal sectors and one of 6 different colours is used to colour each sector. No two adjacent sectors can have the same colour. Find the number of such distinct colourings possible.

Sol. Let for n sectors number of ways be a_n.

With six colours, $a_1 = 6$, $a_2 = 30$

and $a_n + a_{n-1} = 6 \times 5^{n-1}$, $n \geq 3$

$\Rightarrow \quad (-1)^n a_n - (-1)^{n-1} a_{n-1} = (-1)^n 6 \cdot 5^{n-1}$

plug $n = 3, 4, \ldots, n$ and add.

$(-1)^n a_n - a_2 = 6[-5^2 + 5^3 \ldots (-5)^{n-1}]$

$$\cancel{6}(-5)^2 \left[\frac{1 - (-5)^{n-2}}{\cancel{6}}\right]$$

$(-1)^n a_n = 5 + (-5)^n \Rightarrow a_n = 5(-1)^n + 5^n \Rightarrow \boxed{a_{12} = 5^{12} + 5}$