SOLUTION
THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
KAPREKAR CONTEST - FINAL - SUB JUNIOR
CLASS - VII & VIII

Instructions:

1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

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1. (a) In the grid given here, the numbers against each row or column is the sum of all the numbers in that respective row or column.

For example, in the top row, a + b + c + d = 17, etc.

Find the value of a, b, c and d.

a	b	c	d	17
a	c	c	b	19
c	b	a	b	15
d	d	a	b	15
16	18	18	14	

(b) If $\frac{a}{b}+\frac{b}{a}=x$, $\frac{b}{c}+\frac{c}{b}=y$, $\frac{c}{a}+\frac{a}{c}=z$ find the numerical value of $(x^2 + y^2 + z^2 - xyz)$.

Sol. (a) From the given grid :-

$a + b + c + d = 17$ ....(1)

$a + 2c + b = 19$ ....(2)

$a + 2b + c = 15$ ....(3)

$a + b + 2d = 15$ ....(4)

$2a + c + d = 16$ ....(5)

$2b + c + d = 18$ ....(6)

$2a + 2c = 18$ ....(7)

$3b + d = 14$ .....(8)

From equation (7) $a + c = 9$ ....(9)

equation (6) = equation (7)

$2b + c + d = 2a + 2c$

$2a + c = 2b + d$ .....(10)

equation (3) = (4)

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$b + c = 2d$ ....(11)

From equation (9), (10) and (11)

$18 - 2c + c = 2b + d$

$18 - c = 4d - 2c + d$

$18 = 5d - c$ .....(12)

equation (1) – (2)

$d - c = -2$ .....(13)

From equation (12) and (13)

$$\frac{18+c}{5} - c = -2$$

$\boxed{c = 7}$ Put this value in equation (13) will give you $\boxed{d = 5}$

From equation (9) $\boxed{a = 2}$ and from equation (1) $\boxed{b = 3}$

(b) If $\frac{a}{b} + \frac{b}{a} = x$, $\frac{b}{c} + \frac{c}{b} = y$, $\frac{c}{a} + \frac{a}{c} = z$ then

Putting the above values in the equation $(x^2 + y^2 + z^2 - xyz)$

$$\left(\frac{a}{b} + \frac{b}{a}\right)^2 + \left(\frac{b}{c} + \frac{c}{b}\right)^2 + \left(\frac{c}{a} + \frac{a}{c}\right)^2 - \left(\frac{a}{b} + \frac{b}{a}\right) \times \left(\frac{b}{c} + \frac{c}{b}\right) \times \left(\frac{c}{a} + \frac{a}{c}\right)$$

$$\frac{a^2}{b^2} + \frac{b^2}{a^2} + 2 + \frac{b^2}{c^2} + \frac{c^2}{b^2} + 2 + \frac{c^2}{a^2} + \frac{a^2}{c^2} + 2 - \left(\frac{ab}{bc} + \frac{ac}{b^2} + \frac{b^2}{ac} + \frac{bc}{ab}\right)\left(\frac{c}{a} + \frac{a}{c}\right)$$

$$\frac{a^2}{b^2} + \frac{b^2}{a^2} + 2 + \frac{b^2}{c^2} + \frac{c^2}{b^2} + 2 + \frac{c^2}{a^2} + \frac{a^2}{c^2} + 2 - \left(\frac{abc}{abc} + \frac{a^2}{c^2} + \frac{c^2}{b^2} + \frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{abc}{abc}\right)$$

$= 4$

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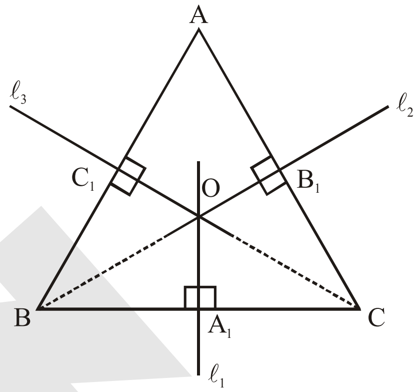
2. ABC is a triangle. ℓ_1, ℓ_2, ℓ_3 are three lines that are perpendicular to the sides BC, CA and AB at A_1, B_1 and C_1 respectively. Prove that l_1, l_2 and l_3 are concurrent if and only if $A_1B^2 + C_1A^2 + B_1C^2 = A_1C^2 + C_1B^2 + B_1A^2$

Sol. Let us first prove if perpendiculars are concurrent then results hold.

Let O be point of concurrency and OA_1, OB_1, OC_1 are drawn perpendicular to the sides BC, CA, AB respectively of a triangle ABC



$BA_1^2 = OB^2 - OA_1^2$

$A_1C^2 = OC^2 - OA_1^2$

$\Rightarrow BA_1^2 - A_1C^2 = OB^2 - OC^2$ ....(1)

Similarly

$CB_1^2 - B_1A^2 = OC^2 - OA^2$ ....(2)

$AC_1^2 - C_1B^2 = OA^2 - OB^2$ ....(3)

Adding equation (1), (2) and (3), we get

$BA_1^2 - A_1C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = OB^2 - OC^2 + OC^2 - OA^2 + OA^2 - OB^2 = 0$

Proof of Converse : If A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC such that $BA_1^2 - A_1C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = 0$, then the perpendiculars at A_1, B_1, C_1 to the respective sides are concurrent.

Proof : Let the perpendiculars at A_1, B_1 to BC, CA respectively meet at O. Let OC_2 be the perpendicular from O to AB

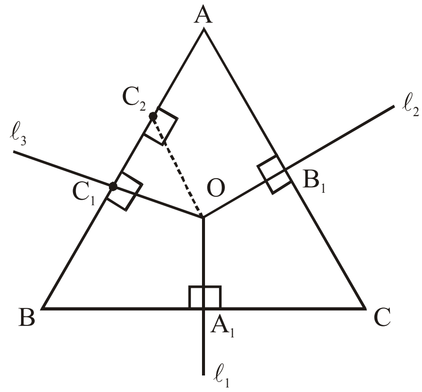
Using previous result :

$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_2A^2 - C_2B^2 = 0$ ....(1)

But it is given that

$BA_1^2 - A_1C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = 0$ ....(2)

$\therefore$ From equation (1) and (2)



$AC_2^2 - C_2B^2 = AC_1^2 - C_1B^2$

$(AC_2 + C_2B)(AC_2 - C_2B) = (AC_1 + C_1B)(AC_1 - C_1B)$

$AB(AC_2 - C_2B) = AB(AC_1 - C_1B)$

$\Rightarrow AC_2 - C_2B = AC_1 - C_1B$ (As $AB \neq 0$)

$\therefore AC_1 - AC_2 = C_1B - C_2B$

$\Rightarrow C_1C_2 = -C_2C_1$

$\Rightarrow 2C_1C_2 = 0$ $\Rightarrow C_1C_2 = 0$

That is, C_1 and C_2 coincide

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3. (a) If a, b are real numbers each greater than 1, prove :

$$(1 + a)(1 + b) < 2(ab + 1).$$

Use this result to prove

$$8(abcd + 1) > (a + 1)(b + 1)(c + 1)(d + 1),$$

where a, b, c, d are real numbers each greater than 1.

(b) The seven digit number 1234*** is a perfect square number, where each * represents a digit (may be same or distinct). Find the square root of this seven digit number.

Sol. (a) Given $a > 1$, $b > 1$

So, $a - 1 > 0$

$b(a - 1) > a - 1$

$ab - b > a - 1$

$1 + ab > a + b$

adding $(ab + 1)$ both side

$2(ab + 1) > a + b + ab + 1$

$\boxed{2(ab+1) > (a+1)(b+1)}$ .....(1)

$a > 1, b > 1, c > 1, d > 1$

So, $ab > 1$, $cd > 1$

Using equation (1) $2(ab\cdot cd + 1) > (ab + 1)(cd + 1)$

Multiplying with 4 both the sides

$8(ab\cdot cd + 1) > 2(ab + 1)\,2(cd + 1)$ .....(2)

Using equation (1)

$2(ab+1) > (a+1)(b+1)$

$2(cd + 1) > (c + 1)(d + 1)$

Multiply the above two equations we will get

$2(ab + 1)\,2(cd + 1) > (a + 1)(b + 1)(c + 1)(d + 1)$

Using equation (2)

$\boxed{8(ab\cdot cd + 1) > (a + 1)(b + 1)(c + 1)(d + 1)}$

(b) The given number is 1234***

The smallest number of this format is 1234000 and

$\sqrt{1234000} = 1110.85$

So, the required 7 digit number should be $(1111)^2 = 1234321$

So, The answer is $\sqrt{1234321} = 1111$

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4. (a) If a, b, c are integers (a, b, c are not zero) such that $a + b + c = 0$, prove that $(2a^4 + 2b^4 + 2c^4)$ is the square of an integer.

(b) Find the smallest positive integer 'n' such that $4^{27} + 4^{1000} + 4^n$ is a perfect square.

Sol. (a) If $a + b + c = 0$ ...... (1)

$2a^4 + 2b^4 + 2c^4$

Replacing c from equation (1)

$2a^4 + 2b^4 + 2(a + b)^4$

$= 2[a^4 + b^4 + a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4]$

$= 4[a^4 + b^4 + 2a^3b + 3a^2b^2 + 2ab^3]$

$= 4[a^4 + a^2b^2 + b^4 + 2(a^2 \cdot ab + a^2b^2 + ab \cdot b^2)]$

$= 4[a^2 + ab + b^2]^2$

$\Rightarrow$ $[2a^2 + 2ab + 2b^2]^2$

(b) $4^{27} + 4^{1000} + 4^n$

$(2^{27})^2 + (2^{1000})^2 + 2^{2n}$

To minimize the value of n and make it a perfect square, let us arragne it in the following way

$(2^{27})^2 + 2^{2n} + (2^{1000})^2$

$\Rightarrow$ $2^{2n} = 2.2^{27}.2^{1000}$

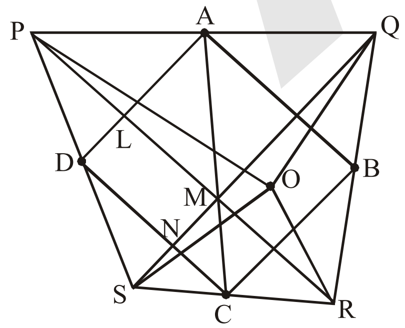
$2^{2n} = 2^{1028}$

$\Rightarrow$ $n = 514$

5. PQRS is a convex quadrilateral.

Point O is inside the quadrilateral such that OP = OQ and OS = OR. If $\angle POQ = \angle SOR = 120°$ and A, B, C are respectively the mid points of PQ, QR and RS, prove that ABC is an equilateral triangle.

Sol.



Let D be the mid-points of PS

A, B, C, D are mid points of PQ, QR, RS, PS

In ΔQOS and ΔPOR

$OS = OR$

$OQ = OP$

$\angle POR = \angle QOS$

So, $\Delta QOS \cong \Delta POR$

By CPCT (1) $QS = PR$, As Using mid point theorem we can say ABCD is rhombus.

(2) $\angle QSO = \angle PRO \Rightarrow$ MORS is cyclic quadrilateral

$\Rightarrow \angle SMR = \angle SOR = 120°$

$\Rightarrow \angle SMP = 60°$ (Using linear pair)

As $MN \| DL$ and $LM \| DN$ So, LMND is a parallelogram so $\angle LDN = \angle LMN = 60°$ and as ABCD is a rhombus so $\angle ABC = \angle LDN = 60°$ and $AB = BC$

$\Rightarrow \angle ABC = \angle BAC = \angle ACB = 60°$

$\Rightarrow \Delta ABC$ is equilateral triangle.