SOLUTION
THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
BHASKARA CONTEST - FINAL - JUNIOR
CLASS - IX & X

Instructions:
1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

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1. (a) If $a^2y^4 + b^2x^4 = a^2b^2$ and $a^2 + b^2 = x^2 + y^2 = 1$, prove that $a^4y^6 + b^4x^6 = (a^2y^4 + b^2x^4)^2$.

(b) Determine all integers m, n such that

$$(m^3 + n)(n^3 + m) = (m + n)^4.$$

Sol. (a) Note we have, $a^2 + b^2 = x^2 + y^2 = 1$

$$\boxed{x^2 - a^2 = b^2 - y^2}$$

we have : $b^2x^4 + a^2y^4 = a^2b^2 \cdot 1$

$= b^2x^4 + a^2y^4 = a^2b^2 (x^2 + y^2)$

$\Rightarrow b^2x^2(x^2 - a^2) = a^2y^2(b^2 - y^2)$

but we have $x^2 - a^2 = b^2 - y^2$

$\therefore \boxed{b^2x^2 = a^2y^2}$

$\Rightarrow b^2x^2 - a^2y^2 = 0$. Squaring both the sides, we get,

$b^4x^4 + a^4y^4 = 2a^2b^2x^2y^2$

Now, $b^4x^6 + a^4y^6 = (b^4x^6 + a^4y^6)(x^2 + y^2)$

$= b^4x^8 + a^4y^8 + x^2y^2(a^4y^4 + b^4x^4)$

$= b^4x^8 + a^4y^8 + 2a^2b^2x^2y^2$

$= (b^2x^4 + a^2y^4)^2$

Hence proved

(b) $(m^3 + n)(n^3 + m) = (m + n)^4$

$\Rightarrow m^3n^3 + \cancel{m^4} + \cancel{n^4} + mn = \cancel{m^4} + 4m^3n + 6m^2n^2 + 4mn^3 + \cancel{n^4}$

$\Rightarrow 4m^3n + 4mn^3 + 6m^2n^2 - m^3n^3 - mn = 0$

Case 1 : If mn = 0

If m = 0, whole expression becomes 0 similiarly if n = 0, whole expression becomes 0.

So, (m, n) = (0, k) ; (ℓ, 0) ; (0, 0) Satisfies, where k & ℓ we any integers.

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Case 2 : If none of m, n = 0 i.e. $(mn \neq 0)$

divide whole expression by mn.

$\Rightarrow 4m^2 + 4n^2 + 6mn - m^2n^2 - 1 = 0$ ....(i)

$\Rightarrow 4(m^2 + n^2 + 2mn) = m^2n^2 + 1 + 2mn$

$\Rightarrow (2m + 2n)^2 - (mn + 1)^2 = 0$

$\therefore$ Final factorisation of eqn. (1) will be

$(mn + 2m + 2n + 1)(mn - 2m - 2n + 1) = 0$

Case A :

$mn + 2m + 2n + 1 = 0$

$m(n + 2) + 2(n + 2) - 3 = 0$

$(m + 2)(n + 2) = 3 = 3 \times 1 \quad -3 \times -1$

$1 \times 3 \quad -1 \times -3$

$m + 2 = 3 \Rightarrow m = 1, n + 2 = 1 \Rightarrow n = -1$

$m + 2 = 1 \Rightarrow m = -1, n + 2 = 3 \Rightarrow n = 1$

$m + 2 = -3 \Rightarrow m = -5, n + 2 = -1 \Rightarrow n = -3$

$m + 2 = -1 \Rightarrow m = -3, n + 2 = -3 \Rightarrow n = -5$

$\therefore$ here (m, n) are

$\boxed{(1, -1) ; (-1, 1) ; (-5, -3) ; (-3, -5)}$

Case B : $mn - 2m - 2n + 1 = 0$

$m(n - 2) - 2(n - 2) - 3 = 0$

$(m - 2)(n - 2) = 3 = 3 \times 1 \quad ; \quad -3 \times -1$

$3 \times 1 \quad ; \quad -3 \times -1$

$m - 2 = 3 \Rightarrow m = 5, n - 2 = 1 \Rightarrow n = 3$

$m - 2 = -3 \Rightarrow m = -1, n - 2 = -1 \Rightarrow n = 1$

$m - 2 = 1 \Rightarrow m = 3, n - 2 = 3 \Rightarrow n = 5$

$m - 2 = -1 \Rightarrow m = 1, n - 2 = -3 \Rightarrow n = -1$

$\therefore$ here (m, n) are

(5, 3) ; (–1, 1) ; (3, 5) ; (1, –1), but here some are repeated.

So final integer pairs (m, n) are

$(0, k) ; (\ell, 0) ; (0, 0), (1, -1) ; (-1, 1) ; (-5, -3) ; (-3, -5) ; (5, 3) ; (3, 5)$.

Here k, ℓ are any integers.

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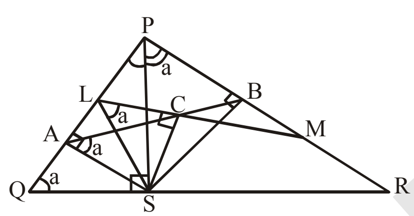
2. In triangle PQR, $\angle P = 90°$. PS is the altitude. From S are drawn SA, SB respectively perpendicular to PQ and PR. C is an arbitrary point on the line segment AB. The line perpendicular to SC meets PQ at L and PR at M.

(i) Prove that the circumcircle of triangle PLM passes through S.

(ii) If C_1 is another arbitary point on the line segment AB and the line perpendicular to SC_1 meets PQ at L_1 and PR at M_1.

Prove that $\dfrac{LL_1}{MM_1}$ is a constant.

Sol. (i)



$PS \perp QR$ $\quad\quad$ $\angle P = 90$

$\therefore$ $\angle SQP = \angle SPB = a$

$\angle PAS + \angle PBS = 90 + 90 = 180$

$\therefore$ PASB is cyclic

$\angle SPB = \angle SAB = a = \angle SPM$ ...(1)

$\angle LAS + \angle LCS = 90 + 90 = 180$

$\therefore$ LASC is cyclic

$\angle SAC = \angle SLC = a = \angle SLM$ ...(2)

$\therefore$ from (1) and (2)

$\angle SPM = \angle SLM = a$

$\therefore$ PLSM is cyclic

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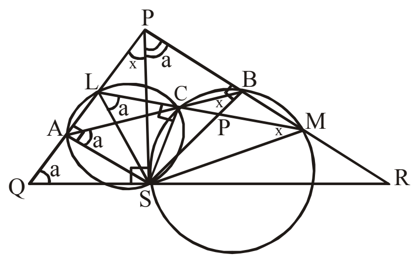
(ii)



$\angle L_1 AS + \angle L_1 C_1 S = 90 + 90 = 180$

L_1ASC_1 is cyclic

$\angle LCS + \angle LAS = 90 + 90 = 180$

LASC is cyclic

$\angle SCM = \angle SBM = 90$

SCBM is cyclic

$\angle SC_1M_1 = \angle SBM_1 = 90$

SC_1BM_1 is cyclic

As PASB is cyclic

$\angle APS = \angle ABS = x$

Also $\angle SCM = \angle SBM = 90$

SCBM is cyclic

$\therefore$ $\angle SBC = \angle SMC = x$

In Δ LSM and ΔASB

$\angle SLM = \angle SAB = a$

$\angle SML = \angle SBA = x$

$\therefore$ $\Delta LSM \sim \Delta ASB$ ...(3)

Similarly

$\Delta L_1SM_1 \sim \Delta ASB$ ...(4)

$\therefore$ From (3) and (4)

$\Delta LSM \sim \Delta L_1SM_1$

$$\therefore \frac{LM}{L_1M_1} = \frac{SM}{SM_1} = \frac{LS}{L_1S}$$

By spiral symmetry

$\frac{LL_1}{MM_1}$ is a constant ratio.

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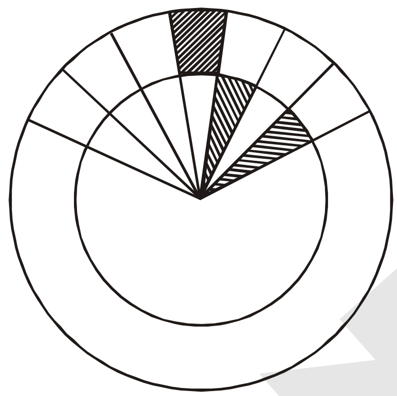
3. (a) Two concentric discs each divided into 100 sectors lie as shown in the diagram. On each disc, fifty sectors are selected at random and painted. The centre disc is rotated to any arbitrary position and the number of coloured regions that match are counted. Show that regardless of the initial colouring, there must be some position that has at least 50 matches.



(b) Three natural numbers form an increasing geometric progression whose sum is 57. Determine the progression.

Sol. (a) Look at one particular sector as the inner disc moves through all 100 posible positions. If will match with an outer sector exactly 50 times. Since this is true for each inner sector, there is a total of $100 \times 50 = 5000$ matches fer the 100 positions. Thus the average number of matches per position is 50, So, there must be atleast one position that has 50 or more matches.

(b) Let the three numbers be a, ar, ar^2.

$a + ar + ar^2 = 57$

$a(1 + r + r^2) = 57 = 1 \times 57 \;;\; 57 \times 1$

$= 3 \times 19 \;;\; 19 \times 3$

Case-1 : $a = 1$, $1 + r + r^2 = 57$

$r^2 + r - 56 = 0 \quad \Rightarrow \quad r^2 + 8r - 7r - 56 = 0$

$r(r + 8) - 7(r + 8) = 0 \quad \Rightarrow \quad (r + 8)(r - 7) = 0$

$\therefore$ $r = 7, -8$, but since natural numbers are given, So, $r = 7$

then numbers will be $a, ar, ar^2 = 1, 7, 49$

Case-2 : $a = 57$, $1 + r + r^2 = 1 \quad \Rightarrow \quad r^2 + r = 0$ not possible

Case-3 : $a = 3$, $1 + r + r^2 = 19 \quad \Rightarrow \quad r^2 + r - 18 = 0$ no integer solution

Case-4 : $a = 19$, $1 + r + r^2 = 3 \quad \Rightarrow \quad r^2 + r - 2 = 0$

$r^2 + 2r - r - 2 = 0$

$r(r + 2) - 1(r + 2) = 0$

$r = 1$, $r = -2$, here $r = -2$ is rejected.

So, $r = 1$, then numbers will be $a, ar, ar^2 \Rightarrow 19, 19, 19$

But since increasing GP is given so it will be rejected. So only solution is 1, 7, 49.

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4. (a) Solve the equation for x : $(a + b + x)^3 - 4(a^3 + b^3 + x^3) = 12abx$.

(b) If a, b, c are positive integers, prove :

$$a^{\frac{a}{a+b+c}}\, b^{\frac{b}{a+b+c}}\, c^{\frac{c}{a+b+c}} \geq \frac{a+b+c}{3}$$

Sol. (a) $(a + b + x)^3 - 4(a^3 + b^3 + x^3) = 12abx$

$\Rightarrow a^3 + b^3 + x^3 + 3(a + b)(b + x)(x + a) - a^3 - b^3 - x^3 - 3(a^3 + b^3 + x^3) = 12abx$

$\Rightarrow 3(a^3 + b^3 + x^3) + 12abx - 3(a + b)(b + x)(x + a) = 0$

$\Rightarrow (a^3 + b^3 + x^3) + 4abx - (a + b)(b + x)(x + a) = 0$

$\Rightarrow a^3 + b^3 + x^3 + 4abx - (abx + a^2b + ax^2 + a^2x + b^2x + ab^2 + bx^2 + abx)$

$\Rightarrow a^3 + b^3 + x^3 + 2abx - a^2b - a^2x - x^2a - x^2b - b^2x - b^2a = 0$

$\Rightarrow x^3 - x^2a - x^2b + a^3 - a^2b - a^2x + b^3 - b^2x - b^2a + 2abx = 0$

$\Rightarrow x^2(x - a - b) - a^2(x - a - b) - b^2(x - a - b) - 2a^2b - 2ab^2 + 2abx = 0$

$\Rightarrow x^2(x - a - b) - a^2(x - a - b) - b^2(x - a - b) + 2ab(x - a - b) = 0$

$\Rightarrow (x - a - b)(x^2 - a^2 - b^2 + 2ab) = 0$

$\Rightarrow x - a - b = 0$	$x^2 - (a^2 + b^2 - ab) = 0$
$\Rightarrow x = a + b$	$x^2 - (a - b)^2 = 0$
	$(x + a - b)(x - a + b) = 0$
	$x = a - b$ or $b - a$

So, overall 3 solutions possible.

$x = a + b,\ x = a - b,\ x = b - a$

(b) by weighted GM $\geq$ weighted HM

$$\Rightarrow (a^a\, b^b\, c^c)^{\frac{1}{a+b+c}} \geq \frac{a+b+c}{\frac{a}{a}+\frac{b}{b}+\frac{c}{c}}$$

$$\Rightarrow a^{\frac{a}{a+b+c}}.b^{\frac{b}{a+b+c}}.c^{\frac{c}{a+b+c}} \geq \frac{a+b+c}{3}$$

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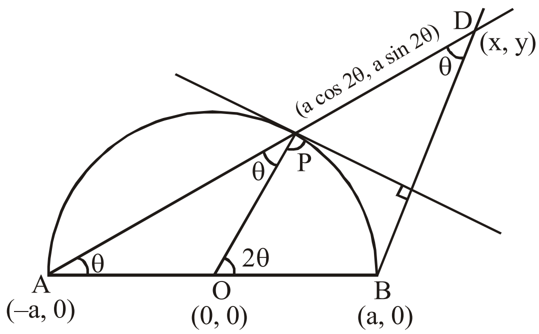
5. AOB is the diameter of a semicircle, with O as the centre. P is a variable point on the semi-circle. The perpendicular from B to the tangent at P and the line AP extended meet at D. Find the locus of D.

Sol.



since AB = BD

and, $\dfrac{AO}{AB} = \dfrac{OP}{BD}$

$\dfrac{a}{2a} = \dfrac{a}{BD}$

BD = 2a

$\Rightarrow$ $(x - a)^2 + (y - 0)^2 = (2a)^2$

$x^2 + y^2 - 2ax + a^2 = 4a^2$

$\Rightarrow$ $x^2 + y^2 - 2xa - 3a^2 = 0$

so, locus is a circle centred at B

and radius equal to the diameter of a given circle.