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INDIAN OLYMPIAD QUALIFIER IN MATHEMATICS - KV (IOQM-KV) 2020-21

Date: 30/01/2021

Max. Marks: 100 | **SOLUTIONS** | **Time allowed: 3 hours**

Instructions:

1. There are 30 questions in this question paper. Total marks is 100.
2. The answer to every question is an integer In the range 00 – 99.
3. Question 1 to 8 carry 2 marks each. Question 9 to 21 carry 3 marks each. Questions 22 to 30 carry 5 marks each.
4. Time allotted is 3 hours.

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1. If a,b,c are real numbers and

$(a + b - 5)^2 + (b + 2c + 3)^2 + (c + 3a - 10)^2 = 0$

find the integer nearest to $a^3 + b^3 + c^3$.

Ans. 57

Sol. As $a, b, c \in R$

$a + b = 5 \Rightarrow a = 5 - b$

$b + 2c = -3 \Rightarrow c = \dfrac{-3-b}{2}$

$c + 3a = 10$

$\dfrac{-3-b}{2} + 3(5-b) = 10$

$-3 - b + 30 - 6b = 20$

$7b = 7 \Rightarrow b = 1 \quad \Rightarrow a = 4 ; c = -2$

$\therefore\ a^3 + b^3 + c^3 = 64 + 1 - 8 = 57$

2. If ABCD is a rectangle and P is a point inside it such that AP = 33, BP = 16, DP = 63. Find CP.

Ans. 56

Sol. Let O is intersection point

Join OP, BO = AO

ΔAPC, OP median

$\therefore\ AP^2 + PC^2 = 2(AO^2 + OP^2)$

(by appolonius theorem)

In ΔBPD

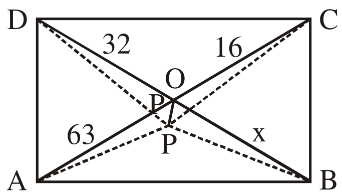
$BP^2 + PD^2 = 2(BO^2 + OP^2)$

Now BO = OA

$AP^2 + PC^2 = BP^2 + PD^2$

$63^2 + 16^2 = x^2 + 33^2$

$\therefore\ x = 56$ cm



3. Sita and Geeta are two sisters. If Sita's age is written after Geeta's age a four digit perfect square (number) is obtained. If the same exercise is repeated after 13 years another four digit perfect square (number) will be obtained. What is the sum of the present ages of Sita and Geeta?

Ans. 55

Sol. Let Sita $\rightarrow$ ab

Geeta $\rightarrow$ cd

cdab $\rightarrow$ perfect square = 100cd + ab = P^2

Also, (cd + 13) (ab + 13) $\rightarrow$ perfect square

$\Rightarrow$ 100cd + 1300 + ab + 13 = Q^2 $\quad\Rightarrow P^2 + 1313 = Q^2$

$\Rightarrow (Q - P)(Q + P) = 1313$

$\Rightarrow (Q - P)(Q + P) = 13 \times 101$

$Q - P = 13$

$Q + P = 101$

Solving we get

$Q = 57$

$P = 44$

$\therefore$ 100cd + ab = P^2 = 1936

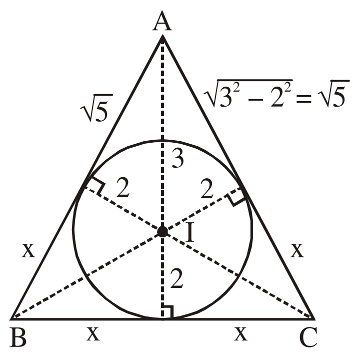
cd = 19 ; ab = 36

$\therefore$ ab + cd = 55

4. Let ABC be an isosceles triangle with AB = AC and incentre I. If AI = 3 and the distance from I to BC is 2, what is the square of the length of BC?

Ans. 80

Sol. $S = \sqrt{5} + 2x$



Area = $\frac{1}{\cancel{2}}(AB + BC + CA)\cancel{2} = \sqrt{\left(2x+\sqrt{5}\right)\left(\sqrt{5}\right)(x)(x)}$

$\left(\sqrt{5}+x\right)+(2x)+\left(\sqrt{5}+x\right)$

$2^2\left(\sqrt{5}+2x\right)^2 = \left(2x+\sqrt{5}\right)\left(\sqrt{5}\right)x^2$

$4\left(\sqrt{5}+2x\right) = \sqrt{5}x^2$

$4\sqrt{5} + 8x = \sqrt{5}x^2$

$\sqrt{5}x^2 - 8x - 4\sqrt{5} = 0$

$x = -\frac{2\sqrt{5}}{5},\ x = 2\sqrt{5}$

$\therefore$ BC = 2x

BC = $2\left(2\sqrt{5}\right)$

BC = $4\sqrt{5}$

$BC^2 = 80$

5. Find the number of positive integers n such that the highest power of 7 dividing n! is 8.

Ans. 07

Sol. n!

$$\therefore \quad \left[\frac{55}{7}\right]=7$$

$$\left[\frac{7}{7}\right]=1$$

$$+$$

$$8$$

$$\left[\frac{49}{7}\right]=7$$

$$\left[\frac{7}{7}\right]=1$$

$$+$$

$$8$$

$\therefore$ from n = 49 to 55

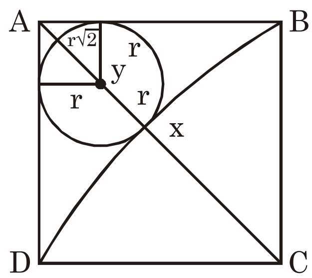
n! will have highest power of 7 as 8

$\therefore$ 7 positive integers possible.

6. Let ABCD be a square with side length 100. A circle with centre C and radius CD is drawn. Another circle of radius r, lying inside ABCD, is drawn to touch this circle externally and such that the circle also touches AB and AD. If $r = m + n\sqrt{k}$. where m, n are integers and k is a prime number, find the value of $\frac{m+n}{k}$.

Ans. 50

Sol. By symmetry, A, Y, X, C are collinear



also, AC = $100\sqrt{2}$

$100\sqrt{2}$ = AC = AY + YX + XC = $r\sqrt{2}$ + r + 100

$$\Rightarrow r=100\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$=100\left(\sqrt{2}-1\right)^2$$

$$=\underset{m}{\underset{\downarrow}{300}}-\underset{n}{\underset{\downarrow}{200}}\underset{k}{\underset{\downarrow}{\sqrt{2}}}$$

$$\Rightarrow \frac{m+n}{k}=\frac{100}{2}=50$$

7. a, b, c are positive real numbers such that $a^2 + b^2 = c^2$ and $ab = c$. Determine the value of

$$\left|\frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{c^2}\right|.$$

Ans. 04

Sol. $$\left|\frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{c^2}\right|$$

$$= \left|\frac{\left\{(a+b)^2-c^2\right\}\left\{c^2-(b-a)^2\right\}}{c^2}\right|$$

$$= \left|\frac{\left(c^2+2ab-c^2\right)\left(c^2-c^2+2ab\right)}{(ab)^2}\right|$$

we have $a^2 + b^2 = c^2$, $ab = c$

$$\left|\frac{4a^2b^2}{a^2b^2}\right| = |4| = 4$$

8. Find the largest 2-digit number N which is divisible by 4, such that all integral power of N end with N.

Ans. 76

Sol. N can be 96, 92, 88, 84 ...

Largest N among then whose all integeral powers of N ends with 9 is '76'

as $76 \times 76 = 5776$

$76 \times 76 \times 76 = 438976$

9. Find the number of ordered triples (x,y,z) of real numbers that satisfy the system of equation

$x + y + z = 7$; $x^2 + y^2 + z^2 = 27$; $xyz = 5$

Ans. 03

Sol. $x + y + z = 7$, $x^2 + y^2 + z^2 = 27$, $xyz = 5$

$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

$\frac{7^2 - 27}{2} = xy + yz + zx$

$11 = xy + yz + zx$

$x + y + z = 7$

$xyz = 5$. Let x, y, z be the roots of

$t^3 - 7t^2 + 11t - 5 = 0 \rightarrow (t-1)^2 (t-5) = 0$

$\therefore$ t = 1, 5, 1

$\therefore (x, y, z) \rightarrow (1, 1, 5)$

$(1, 5, 1)$

$(5, 1, 1)$

3 pairs possible

10. Let A and B be two finite sets such that there are exactly 144 sets which are subsets of A or subsets of B. Find the number of elements in $A \cup B$.

Ans. 08

Sol. $\left.\begin{array}{l}\text{Let A contain x elements}\\ \text{\& B contain y elements}\end{array}\right]$ common elements z

total subsets of A $\rightarrow 2^x$ $\quad \begin{pmatrix} x > z \\ y > z \end{pmatrix}$

total subsets of B $\rightarrow 2^y$

common subsets of A & B $\rightarrow 2^z$

$\therefore\ 2^x + 2^y - 2^z = 144$

$2^7 + 2^5 - 2^4 = 144 \rightarrow$ only possible

$\therefore\ n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= x + y - z$

$= 7 + 5 - 4 = 8$

11. The prime numbers a, b and c are such that $a + b^2 = 4c^2$. Determine the sum of all possible values of $a + b + c$.

Ans. 31

Sol. $a, b, c \in$ prime number

$(2c)^2 - (b)^2 = a$

$(2c + b)\ (2c - b) = a$

a	1
1	a
–a	–1
–1	–a

Case 1 :

$2c + b = a \qquad 2\left(\frac{a+1}{4}\right) + b = a$

$2c - b = 1$

$4c = a + 1 \qquad b = a - \frac{a+1}{2} = \frac{2a-a-1}{2}$

$c = \frac{a+1}{4} \qquad b = \frac{a-1}{2}$

Case 2 :

$2c + b = 1$

this equation is not possible as b & c are primes ≥ 2

Case 3 :

$2c + b = -a$

Not possible as LHS is positive but RHS is negative

Case 4 :

$2c + b = -1$

Not possible due to above reason.

So only possible a, b, c is a, $\frac{a-1}{2}, \frac{a+1}{4}$

Now $a + b + c = a + \frac{a-1}{2} + \frac{a+1}{4} = \frac{4a+2a-2+a+1}{4}$

$7a - 1 = 4k$ $\qquad = \frac{7a-1}{4} = k(k \in I)$

$a = \frac{4k+1}{7}$

Put k = 5, we will get a = 3

But $b = \frac{a-1}{2} = \frac{3-1}{2} = \frac{2}{2} = 1 \neq \text{prime}$

So k = 5 is neglected

Put k = 12, we will get a = 7,

$b = \frac{7-1}{2} = 3$, $c = \frac{7+1}{4} = 2$ which are all primes; so a + b + c = k = 12 is a solution.

put k = 19, we will get a = 11

$b = \frac{11-1}{2} = 5$, $c = \frac{11+1}{4} = 3$ which are all primes; so a + b + c = k = 19 is also a solution

checking k = 20, 33, 40, 47, ... be are not getting a, b, c are primes altogether.

$\therefore$ Ans = 12 + 19

= 31

12. Let A = {m : m an integer and the roots of $x^2 + mx + 2020 = 0$ are positive integers.)

and

B = {n : n an integer and the roots of $x^2 + 2020x + n = 0$ are negative integers).

Suppose a is the largest element of A and b is the smallest element of B. Find the sum of digits of a + b.

Ans. 26

Sol. $x^2 + mx + 2020 = 0 \qquad m \in I$

$\swarrow \searrow$

$\alpha \quad \beta$; $\qquad \alpha, \beta \in I^+$

$x^2 + 2020x + n = 0 \qquad n \in I$

$\swarrow \searrow$

$\gamma \quad \delta$; $\qquad \gamma, \delta \in I^-$

we have to find largest m & smallest n

$\alpha + \beta = -m, \qquad \alpha\beta = 2020$

$\because$ LHS is positive so RHS must also be +ve

so m must be –ve. $\qquad$ so $m = -(\alpha + \beta)$ must be I^-

$\gamma + \delta = -2020, \quad \gamma\delta = n$

so $n \in I^+$ γ, δ are negative so there

product will become +ve

and hence n will become +ve

Now,

$\alpha\beta = 2020 = 2^2 \times 5 \times 101 = 2020 \times 1$

Now $\qquad 1010 \times 2$

$\because$ $M = -(\alpha + \beta)$ to $\qquad 505 \times 4$

be maximised, we can $\qquad 404 \times 5$

take (101, 20) to be (α, β) $\qquad 202 \times 10$

so $m = -(101 + 20)$ $\qquad 101 \times 20$

$= -121$

Now $n = \gamma\delta$ and $\gamma + \delta = -2020$

so (γ, δ) can be $(-2019, -1), (-2018, -2), \ldots (-1010, -1010), \ldots (-1, -2019)$

Now smallest n can come when (γ, δ) is $(-2019, -1)$

$\therefore n = -2019 \times -1 = 2019$

$\therefore a = -121, b = 2019$

$a + b = -121 + 2019$

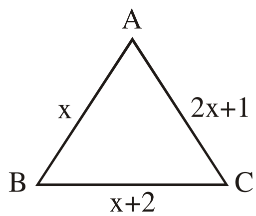
$= 1898$

Sum of digits = 1 + 8 + 9 + 8 = 26

13. The sides of a triangle arc x, 2x + 1 and x + 2 for some positive rational number x. If one angle of the triangle is 60°, what is the perimeter of the triangle?

Ans. 09

Sol. Let $\angle A = 60° \Rightarrow \cos 60 = \dfrac{(x)^2 + (2x+1)^2 - (x+2)^2}{2(x)(2x+1)}$



$$\Rightarrow \frac{1}{2} = \frac{x^2 + 4x^2 + 1 + 4x - x^2 - 4 - 4x}{24x^2 + 2x}$$

$\Rightarrow 2x^2 + x = 4x^2 - 3$

$\Rightarrow 2x^2 - x - 3 = 0$

$$x = \frac{1 \pm \sqrt{1+24}}{4} = \frac{1 \pm 5}{4} = \frac{1+5}{4} \text{ (only)}$$

$$= \frac{3}{2}$$

$\therefore$ perimeter of ΔABC

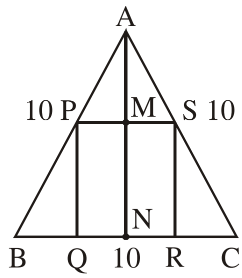
$= AB + BC + AC = 4x + 3$

$$= 4\left(\frac{3}{2}\right) + 3 = 9$$

14. Let ABC be an equilateral triangle with side length 10. A square PQRS is inscribed in it, with P on AB. Q, R on BC and S on AC. If the area of the square PQRS is $m+n\sqrt{k}$ where m, n are integers and k is a prime number then determine the value of $\sqrt{\frac{m+n}{k^2}}$

Ans. 10

Sol. As we know height of equilateral $\Delta = \frac{\sqrt{3}}{2}\times 10$



$AN = 5\sqrt{3}$

Now, $\Delta APS \sim \Delta ABC$

So $\frac{AM}{AN}=\frac{AP}{AB}=\frac{PS}{BC}=\frac{AS}{AC}$

$\frac{AN-MN}{AN}=1-\frac{MN}{5\sqrt{3}}=\frac{PS}{10}$

$\Rightarrow 1-\frac{PS}{5\sqrt{3}}=\frac{PS}{10}$ [MN = PS side of square]

$\Rightarrow 1=PS\left[\frac{1}{10}+\frac{1}{5\sqrt{3}}\right]$

$\Rightarrow 1=PS\left[\frac{\sqrt{3}+2}{10\sqrt{3}}\right]$

$\Rightarrow \frac{10\sqrt{3}}{\sqrt{3}+2}=PS$

$\Rightarrow \frac{\left(10\sqrt{3}\right)\left(2-\sqrt{3}\right)}{\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)}$

$\Rightarrow \frac{20\sqrt{3}-10\times\sqrt{3}\times\sqrt{3}}{4-3}$

$\Rightarrow 20\sqrt{3}-30$

$\Rightarrow PS=10\left(2\sqrt{3}-3\right)$

Hence area of square = PS^2

$=\left(20\sqrt{3}-30\right)^2$

$\Rightarrow 1200+900-1200\sqrt{3}$

$\Rightarrow 2100-1200\sqrt{3}$

on comparing with $m+n\sqrt{k}$

$\Rightarrow m = 2100,\ n = -1200$

$k = 3$

$\Rightarrow \sqrt{\frac{m+n}{k^2}}=\sqrt{\frac{2100-1200}{3^2}}=\sqrt{\frac{900}{9}}=\sqrt{100}=10$

15. Ria has 4 green marbles and 8 red marbles. She arranges them in a circle randomly, if the probability that no two green marbles are adjacent is $\frac{p}{q}$ where the positive integers p,q have no common factors other than 1, what is p + q?

Ans. 40

Sol. 4G, 8R

Total ways = (12 – 1)! = 11!

number of ways in which no two green marbles are together = $7!\,{}^8P_4$

probability = $\frac{7!\,{}^8P_4}{11!}$

$= \frac{8\times7\times6\times5}{11\times10\times9\times8} = \frac{7}{33}$

$\Rightarrow p + q = 40$

16. If x and y are positive integers such that $(x-4)(x-10) = 2^y$, find the maximum possible value of x + y.

Ans. 16

Sol. $(x-4)(x-10) = 2^y$

Let $2^y = 2^a\cdot2^b$

$\therefore\ (x-4)(x-10) = 2^a\cdot2^b$

Let $x-4 = 2^a$

$x-10 = 2^b$

Add

$(2x-14 = 2^a + 2^b)$ ...(i)

Subtract both

$6 = 2^a - 2^b$

$\Rightarrow$ a = 3, b = 1 (only value)

$\therefore\ 2x - 14 = 2^3 + 2^1$

$2x - 14 = 10$

$2x = 24$

$x = 12$

$\therefore 2^y = 2^a\cdot2^b$

$2^y = 2^3 \times 2^1$

$2^y = 2^4$

$y = 4$

$\therefore\ x + y = 12 + 4 = 16$

17. Two sides of a regular polygon with n sides, when extended, meet at an angle of 28°. What is the smallest possible value of n ?

Ans. 45

Sol.. For 28° angle, we need a turn of 28° + 180°

For 208° turn, say we turned K times. In n-gon, we know that one turn in n-gon is $\left(\frac{360}{n}\right)^{\circ}$

$\Rightarrow \left(\frac{360}{n}\right)k = 208$

$\Rightarrow \frac{k}{n} = \frac{208}{360} = \frac{26}{45}$

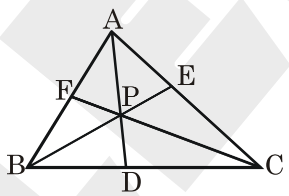
as K, n $\in Z^+$ and $\frac{26}{45}$ is most reduced

we see that n = 45

18. Let D,E,F be points on the sides BC,CA,AB of a triangle ABC, respectively. Suppose AD, BE,CF are concurrent at P. If PF/PC = 2/3, PE/PB = 2/7 and PD/PA = m/n, where m,n are positive integers with gcd(m,n) = 1. find m + n.

Ans. 45

Sol. $\frac{PF}{PC} = \frac{2}{3}, \frac{PE}{PB} = \frac{2}{7}$

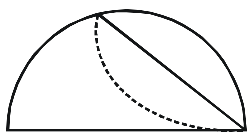


$\frac{PF}{CF} + \frac{PE}{BE} + \frac{PD}{AD} = 1$

$\Rightarrow \frac{2}{5} + \frac{2}{9} + \frac{PD}{AD} = 1 \Rightarrow \frac{PD}{AD} = \frac{17}{45}$

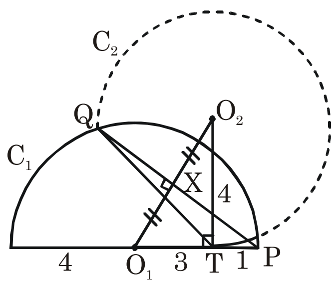
$\frac{PD}{PA} = \frac{17}{28} \Rightarrow m + n = 45$

19. A semicircular paper is folded along a chord such that the folded circular arc is tangent to the diameter of the semicircle. The radius of the semicircle is 4 units and the point of tangency divides the diameter in the ratio 7 : 1. If the length of the crease (the dotted line segment in the figure) is l then determine l^2.



Ans. 39

Sol. We see that O_2 is reflection of O_1 over X, as C_1, C_2 have same radius and the same chord is equidistant from both centres.



Also, $O_2T = r = 4 \Rightarrow O_1O_2 = 5 \Rightarrow O_2X = \frac{5}{4}$

$\Rightarrow PX^2 = O_2P^2 - O_2X^2 = 16 - \frac{25}{4} = 9.75$

$\Rightarrow PQ^2 = 4PX^2 = 39$

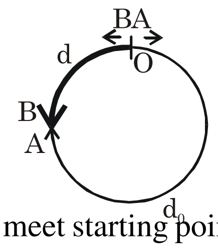
20. Two people A and B start from the same place at the same time to travel around a circulartrack of length 100m in opposite directions. First B goes more slowly than A until they meet, then by doubling his rate he next meets A at the starting point. Let d m be the distance travelled by B before he met A for the first time after leaving the starting point. Find ihe Integer closest to d.

Ans. 41

Sol. Let speed of A is S_0 & distance covered is d_0 and speed of B is S_1 & distance covered is d

As given $S_1 > S_2$

so when they I[st] meet time is same



Hence, $\frac{d_0}{S_0} = \frac{d}{S_1} \Rightarrow \frac{d_0}{d} = \frac{S_0}{S_1}$ ...(1)

so after meeting I[st] time. B double his speed then they meet starting point "O"

If means now "A" cover "d" distance

then "B" cover "d_0" distance

in same time, So

$\frac{d}{S_0} = \frac{d_0}{2S_1} \Rightarrow \frac{d}{d_0} = \frac{S_0}{2S_1}$ ...(2)

Now from (1) equation put in (2)

$\frac{d}{d_0} = \frac{d_0}{2d} \Rightarrow 2d^2 = d_0^{\,2} \Rightarrow 2 = \frac{d_0^{\,2}}{d^2}$

square root both sides

$\Rightarrow \frac{\sqrt{2}}{1} = \frac{d_0}{d}$

so As given total distance 100 i.e. $d_0 + d$

so $d = \frac{1}{\sqrt{2}+1} \times 100 = \left(\sqrt{2}-1\right) \times 100$

$141.4 - 100 = 41.4$ so ≈ 41 m

21. Let A = {1,2,3,4,5,6,7,8}, B = {9,10,11,12,13,14,15,16} and C = {17,18,19,20,21,22,23,24}. Find the number of triples (x,y,z) such that $x \in A$, $y \in B$. $z \in C$ and $x + y + z = 36$.

Ans. 46

Sol. $A = \{1, 2, 3, ..., 8\}$, $B = \{9, 10, ..., 16\}$;

$C = \{17, 18, 19, ..., 24\}$

$x + y + z = 36$

$1 \le x \le 8,\ 9 \le y \le 16,\ 17 \le z \le 24$

$y' = y - 8,\ \ z' = z - 16$

$x + y' + 8 + z' + 16 = 36$

$\Rightarrow x + y' + z' = 12$

Now $x, y', z' \in \{1, 2, 3, ..., 8\}$

Total natural sol = ${}^{12-1}C_{3-1} = {}^{11}C_2 = 55$

$\left.\begin{array}{l} \text{If } x = 9 \Rightarrow y' + z' = 3 \quad \rightarrow 2 \text{ sol} \\ \text{If } x = 10 \Rightarrow y' + z' = 2 \ \rightarrow 1 \text{ sol} \end{array}\right\}\text{Reject}$

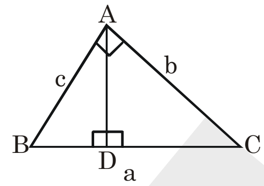
$\therefore$ Reject 3(3) = 9 sol

$\therefore$ Required sol = 55 – 9 = 46

22. Let ABC be a triangle with $\angle BAC = 90°$ and D be the point on the side BC such that $AD \perp BC$. Let r, r_1 and r_2 be the inradii of triangles ABC, ABD, and ACD respectively. If r, r_1, and r_2 are positive integers and one of them is 5. find the largest possible value of $r + r_1 + r_2$.

Ans. **30**

Sol. $\frac{r_1}{r} = \frac{c}{a}, \frac{r_2}{r} = \frac{b}{a}$



$\Rightarrow r_1^2 + r_2^2 = r^2$

one of r, r_1, r_2 = 5

$\Rightarrow (r + r_1 + r_2)_{max} = 5 + 12 + 13 = 30$

23. Find the largest positive integer N such that the number of integers in the set {1,2,3,...,N} which are divisible by 3 is equal to the number of integers which are divisible by 5 or 7 (or both).

Ans. **65**

Sol. {1, 2, 3, ..., N}

No. of integers divisible by 3 = $\left[\frac{N}{3}\right]$

No. of integers divisible by 5 = $\left[\frac{N}{5}\right]$

No. of integers divisible by 7 = $\left[\frac{N}{7}\right]$

No. of integers divisible by 35 = $\left[\frac{N}{35}\right]$

$\therefore n(5 \cup 7) = n(5) + n(7) - n(5 \cap 7)$

$$= \left[\frac{N}{5}\right] + \left[\frac{N}{7}\right] - \left[\frac{N}{35}\right]$$

Now, $\left[\frac{N}{3}\right] = \left[\frac{N}{5}\right] + \left[\frac{N}{7}\right] - \left[\frac{N}{35}\right]$

$$\Rightarrow \left[\frac{N}{35}\right] + \left[\frac{N}{3}\right] = \left[\frac{N}{5}\right] + \left[\frac{N}{7}\right]$$

checking for certain values of N,

will get N = 65 is the largest possibility

$\therefore$ N = 65

24. Two circles, S_1 and S_2, of radii 6 units and 3 units respectively, are tangent to each other, externally. Let AC and BD be their direct common tangents with A and B on S_1, and C and D on S_2. Find the area of quadrilateral ABDC to the nearest integer.

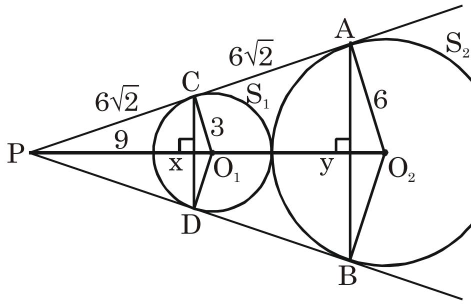
Ans. 68

Sol. We see that CD is polar of P in S_1

We see that AB is polar of P in S_2

$\Rightarrow CD \perp PX,\ AB \perp PY$

$\Rightarrow AB \parallel CD$



Also, $O_1X.O_1D = r_{S_1}^{\ 2} = 9$

$\Rightarrow\ O_1X = \dfrac{9}{O_1P} = \dfrac{9}{q} = 1$

$\Rightarrow O_2Y = 2$ (under similarity)

$\Rightarrow XY = O_1O_2 - O_2Y + O_1X = 9 - 2 + 1 = 8$

$\Rightarrow [ABCD] = \dfrac{1}{2}[AB + CD].XY$

$= \dfrac{1}{2} \times 12\sqrt{2} \times 8^4$

$= 48\sqrt{2} \approx 68$

25. A five digit number $n = \overline{abcde}$ is such that when divided respectively by 2,3,4,5,6 the remainders are a,b,c,d,e. What is the remainder when n is divided by 100?

Ans. 11

Sol. $n = \overline{abcde}$

$n \equiv a(\text{mod } 2) \equiv b(\text{mod } 3) \equiv c(\text{mod } 4) \equiv d(\text{mod} 5) \equiv e(\text{mod } 6)$

Possible remainders mod 2 = 0 or 1

But $a \neq 0$, so a = 1

Now since $n \not\equiv 0$ (mod 2), so its units digit must be 1, 3, 5, 7 or 9.

So $e \in \{1, 3, 5, 7, 9\}$ but since 'e' is the remainder obtained on dividing n by 6, so it will be less than 6.

So $e \in \{1, 3, 5\}$

So $n = \overline{1\,b\,c\,d\,1}, \overline{1\,b\,c\,d\,3}, \overline{1\,b\,c\,d\,5}$

Now, d is the remainder mod 5, so it will be 1, 3, 0 respectively

so $n = \overline{1\,b\,c\,1\,1},\ \overline{1\,b\,c\,3\,3},\ \overline{1\,b\,c\,0\,5}$

Now, c is the remainder mod 4 and any number mod 4 is equivalent to last two digits mod 4.

So it will be 3, 1 and 0 respectively.

So $n = \overline{1\,\_\,3\,1\,1},\ \overline{1\,\_\,1\,3\,3},\ \overline{1\,\_\,0\,0\,5}$

Now this vacant space is b(mod 3), which can be 0, 1 or 2.

If $n = \overline{1\,\_\,3\,1\,1}$	$n = \overline{1\,\_\,1\,3\,3}$	$n = \overline{1\,\_\,0\,0\,5}$
10311 ×	10133 ×	10005 ×
11311 ✓	11311 ×	11005 ×
12311 ×	12311 ×	12005 ×

So the number is n = 11311

and $11311 \equiv 11$ (mod 100)

$\therefore$ Ans = 11

26. Let a,b,c be three distinct positive integers such that the sum of any two of them is a perfect square and having minimal sum a + b + c. Find this sum.

Ans. 55

Sol. $a, b, c \in I^+$

$a + b = p^2$ .... (1)

$b + c = q^2$ .... (2)

$c + a = r^2$ .... (3)

(1) + (2) + (3)

$$a + b + c = \frac{p^2 + q^2 + r^2}{2}$$

Now, since a + b + c is an integer. so either p, q, r must be all even or 1 even and 2 odd.

Now, since a, b, c are all distinct and we went to minimize a + b + c, we went gcd (p, q, r) = 1. Which inturn means that 1 even and 2 odd term are required.

Checking several perfect squares, we get p^2, q^2, r^2 to be 25, 36 and 49.

i.e. $a + b = 25$

$b + c = 36$

$c + a = 49$

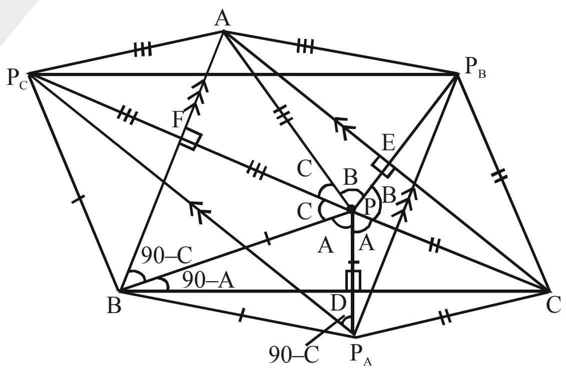
which on solving gives a = 30, b = 19, c = 6

and $(a + b + c)_{min} = 30 + 19 + 6 = 55$

27. Let ABC be an acute-angled triangle and P be a point in its interior. Let P_A, P_B and P_C be the images of P under reflection in the sides BC, CA, and AB, respectively. If P is the orthocentre of the triangle $P_AP_BP_C$ and if the largest angle of the triangle that can be formed by the line segments PA, PB, and PC is x°, determine the value of x.

Ans. 60

Sol. Let



PP_A cut BC at D

PP_B cut AC at E

PP_C cut AB at F

BC is perpendicular bisector of $PP_A \Rightarrow BP = BP_A$

|||y $BP = BP_A = BP_C$, $CP = CP_A = CP_B$,

$AP_C = AP_B = AP$

B is centre of circumcircle of ΔPP_AP_C

$$\Rightarrow \angle PP_AP_C = \frac{1}{2}\angle PBP_C$$

$\angle PBA = \angle ABP_C = \angle PP_AP_C = 90° - \angle C$

$\angle FPB = \angle C$

|||y $\angle PP_CP_A = 90° - \angle A = \angle PBD$

$\Rightarrow \angle BPD = \angle A$

|||y C is cirumcentre of ΔPP_AP_B

and A is circumcentre of ΔPP_BP_C

and |||y $\angle DPC = \angle A$

$\angle CPE = \angle EPA = \angle B$

and $\angle APF = \angle C$

Now $\Delta PBD \cong \Delta PDC$ by ASA

$\Rightarrow PB = PC$

|||y $PC = PA$, $PA = PB$

i.e. P is the circumcentre of ΔABC

$\therefore$ The Δ formed with PA,PB,PC

as sides is equilateral

$\Rightarrow$ Each angle is 60º.

28. For a natural number n, let n' denote the number obtained by deleting zero digits, if any. (For example, if n = 260, n' = 26; if n = 2020, n' = 22.) Find the number of 3-digit numbers n for which n' is a divisor of n. different from n.

Ans. 93

Sol. let start with checking for certain waves fo n.

n = 100,	n' = 1	✓
n = 101,	n' = 11	×
n = 102,	n' = 12	×
n = 103,	n' = 13	×
n = 104,	n' = 14	×
n = 105,	n' = 15	✓
n = 106,	n' = 16	×
n = 107,	n' = 17	×
n = 108,	n' = 18	✓
n = 109,	n' = 19	×
n = 110,	n' = 11	✓
n = 120,	n' = 12	✓
n = 130,	n' = 13	✓
n = 140,	n' = 14	✓
n = 150,	n' = 15	✓
n = 160,	n' = 16	✓
n = 170,	n' = 17	✓
n = 180,	n' = 18	✓
n = 190,	n' = 19	✓
n = 200,	n' = 2	✓

so with this, we abserve that there are two cases possible

Case 1: only zero digits in the end.

In this case n' always divides n. and there are total $9 \times 10 = 90$ 3 digit numbers of this terms.

Case 2 : When 0 is in the ten's place.

ie. $\overline{a0b}$ term

In this form $\overline{ab} | \overline{a0b}$

i.e. $10a + b \mid 100a + b$

i.e. $10a + b \mid 90a$.

Now checking for values of a,b {1,2,....9}

be will get a.b to be (1,5) ; (1,8) ; (4,5)

so total 3 digit numbers is 90 + 3 = 93

29. Consider a permutation (a_1,a_2,a_3,a_4,a_5) of {1,2,3,4,5}. We say the 5-tuple (a_1,a_2,a_3,a_4,a_5) is flawless if for all $1 \le i < j < k \le 5$, the sequence (a_i, a_j, a_k) is **not** an arithmetic progression (in that order). Find the number of flawless 5-tuples.

Ans. 20

Sol. (a_1,a_2,a_3,a_4,a_5)

$\forall\ 1 \le i < j < k \le 5$, (a_i,a_j,a_k) is not as A.P.

$\because$ a_i,a_j,a_k is not an A.P.

(1) 1,5 and 2,4 must be on same side of 3.

(2) 1,3 must be on same side of 2

(3) 3,5 must be on same side of 4.

$\overline{\text{I}}\ \overline{\text{II}}\ \overline{\text{III}}\ \overline{\text{IV}}\ \overline{\text{V}}$

$1 \Rightarrow$ 3 must be on places I, III or V

Case -1 : If 3 is on position III

_ _ $\underline{3}$ _ _ 2!2!2! = 8 ways.

15 2,4

Case 2 : If 3 is on position I

$\underline{3}$ _ _ _ _ 1 must come before 2 and 5 must come before 4.

$\Rightarrow$ No of ways $= \dfrac{4!}{2!2!} = 6$

Case 3 : If 3 is on position V

_ _ _ _ $\underline{3}$ no of ways = 6

$\therefore$ No of flawless

permutations = 8 + 6 + 6 = 20.

30. Ari chooses 7 balls at random from n balls numbered 1 to n. If the probability that no two of the drawn balls have consecutive numbers equals the probability of exactly one pair of consecutive numbers in the chosen balls, find n.

Ans. 54

Sol. No of ways in which no two balls are consecutive = ${}^{n-7+1}C_7 = {}^{n-6}C_7$

To find no of ways in which exactly one pair of consecutive balls are there, Let x_0 be no of balls before 1^{st} selected ball.

$x_i(i = 1 \text{ to } 6)$ denotes no of balls between i^{th} and $(i+1)^{th}$ balls selected and x_7 be no of balls after 7^{th} ball is selected.

Now $x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = n - 7$

Case 1 : If 1^{st} and 2^{nd} selected balls are consecutive then $x_1 = 0$

$\Rightarrow x_0 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = n - 7$

$\geq 0 \;\; \geq 1 \;\; \geq 1 \;\; \geq 1 \;\; \geq 1 \;\; \geq 1 \;\; \geq 0$

$\Rightarrow (x_0 + 1) + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + (x_7 + 1) = n - 5$

$\Rightarrow$ No of solutions = ${}^{n-5-1}C_{7-1} = {}^{n-6}C_6$

|||y If $x_2 = 0$, then ${}^{n-6}C_6$

$x_3 = 0$, then ${}^{n-6}C_6$

$x_6 = 0$, then ${}^{n-6}C_6$

so total ways in which exactly one pair is consecutive = $6({}^{n-6}C_6)$

Now ${}^{n-6}C_7 = 6.\,{}^{n-6}C_6$

$$\Rightarrow \frac{(n-6)!}{7!(n-13)!} = 6.\frac{(n-6)!}{6!(n-12)!}$$

$$\Rightarrow n - 12 = 42 \Rightarrow n = 54$$