INDIAN OLYMPIAD QUALIFIER IN MATHEMATICS (IOQM) - 2020-21

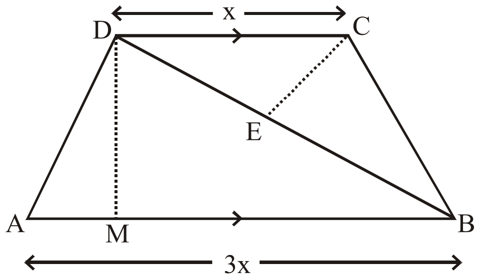
Date: 17/01/2021

Max. Marks: 100 **SOLUTIONS** **Time allowed: 3 hours**

1. Let ABCD be a trapezium in which AB || CD and AB = 3CD. Let E be then: midpoint of the diagonal BD. If [ABCD] = n × [CDE], what is the value of n? (Here [τ] denotes the area of the geometrical figure τ.

Ans. **(08)**

Sol.



[ABCA] = [ABD] + [BCA]

$$= \frac{3x}{2}.H + \frac{x.H}{2}$$

$$[ABCD] = \frac{4x.H}{2} = 2x.H$$

Also, $$[CDE] = \frac{1}{2}[BDC]$$

$$= \frac{1}{2}.\frac{1}{2}xH$$

$$= \frac{1}{4}xH$$

$$\therefore \quad \frac{[ABCD]}{[CDE]} = 8$$

[ABCD] = 8 × [CDE]

So, $\boxed{n = 8}$

2. A number N in base 10, is 503 in base b and 305 in base b + 2. What is the product of the digits of N?

Ans. **(64)**

Sol. $(N)_{10} = (503)_6 = (305)_{6+2}$

$\therefore$ $N = 5b^2 + 3 = 3(b + 2)^2 + 5$

$\therefore$ $5b^2 - 2 = 3(b^2 + 4 + 4b)$

$\Rightarrow$ $2b^2 - 12b - 14 = 0$

$\Rightarrow$ $b^2 - 6b - 7 = 0$

$\Rightarrow$ $(b - 7)(b + 1) = 0$

$\Rightarrow$ $\boxed{b = 7}$

$\therefore$ N = 5(49) + 3

= 248

$\therefore$ Products of digits of N = 2 × 4 × 8

= 64

3. If $\sum_{k=1}^{N} \frac{2k+1}{(k^2+k)^2}$ = 0.9999 then determine the value of N.

Ans. (99)

Sol. $\sum_{k=1}^{N} \frac{2k+1}{\left(k^2+k\right)^2} = 0.9999$

$\Rightarrow \sum_{k=1}^{N} \frac{k+k+1}{k^2\left(k+2\right)^2} = 0.9999$

$\Rightarrow \sum_{k=1}^{N} \frac{1}{k\left(k+1\right)}\left[\frac{1}{k}+\frac{1}{k+1}\right]$

$\Rightarrow \sum_{k=1}^{N} \left[\frac{1}{k}-\frac{1}{k+1}\right]\left[\frac{1}{k}+\frac{1}{k+1}\right] = 0.9999$

$\Rightarrow \sum_{k=1}^{N} \left(\frac{1}{k^2}-\frac{1}{\left(k+1\right)^2}\right) = 0.9999$

$\Rightarrow 1-\frac{1}{\left(N+1\right)^2} = 0.9999$

$\Rightarrow \frac{1}{\left(N+1\right)^2} = 0.0001$

= $(N + 1)^2$ = 10,000

$\Rightarrow$ N + 1 = 100

$\Rightarrow \boxed{N = 99}$

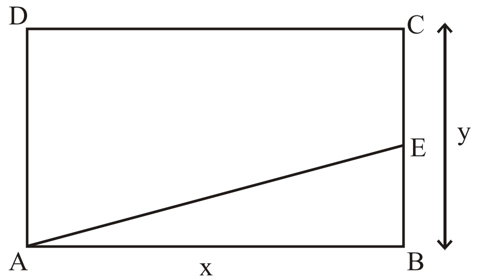
4. Let ABCD be a rectangle in which AB + BC + CD = 20 and AE = 9 where E is the mid-point of the side BC. Find the area of the rectangle.

Ans. (19)

Sol. Let AB = CD = x

and AD = BC = y

Then,



$2x + y = 20, \quad x^2 + \frac{y^2}{4} = 81$ ..... (1)

$\Rightarrow x^2 + \frac{\left(20-2x\right)^2}{4} = 81$

$\Rightarrow x^2 + 100 + x^2 - 20x = 81$

$\Rightarrow 2x^2 - 20x + 19 = 0$

$\Rightarrow \therefore x = \frac{20 \pm \sqrt{400-152}}{4}$

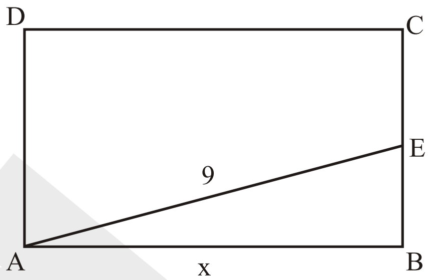
$$= \frac{20 \pm 2\sqrt{62}}{4}$$

$$x = \frac{10 \pm \sqrt{62}}{2}$$

$$\therefore \quad y = 10 \mp \sqrt{62}$$

So, Area of rectagle $= \dfrac{(10)^2 - 62}{2}$ = 19

Method II :



x + y + x = 20

2x + y = 20

$$x + \frac{y}{2} = 10 \qquad \text{..... (1)}$$

$$x^2 + \frac{y^2}{4} = 81 \qquad \text{..... (2)}$$

Square Both side E_C (1)

$$\left(x + \frac{y}{2}\right)^2 = 100$$

$$x^2 + \frac{y^2}{4} + xy = 100$$

xy = 19

5. Find the number of integer solutions to |x| – 2020| < 5.

Ans. (18)

Sol. $\left||x| - 2020\right| < 5$

– 5 < |x| – 2020 < 5

$\Rightarrow$ 2015 < |x| < 2025

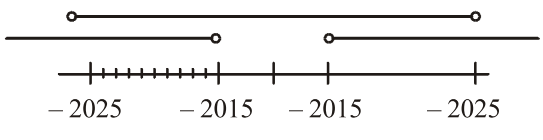
So, |x| > 2015

$\Rightarrow$ – 2015 > x > 2015 ...(1)

Also, |x| < 2025

$\Rightarrow$ – 2025 < x < 2025 ...(2)

From (1) and (2)



So, x ∈ {– 2024, – 2023, ... 2015, 2016, 2017, ...2024}

No of integer solution = 18 Ans.

6. What is the least positive integer by which $2^5 \cdot 3^6 \cdot 4^3 \cdot 5^3 \cdot 6^7$ should be multiplied so that, the product is a perfect square?

Ans. (15)

Sol. Let $M = 2^5 \cdot 3^6 \cdot 4^3 \cdot 5^3 \cdot 6^7$

$\Rightarrow \quad M = 2^{5+6+7} \cdot 3^{6+7} \cdot 5^3$

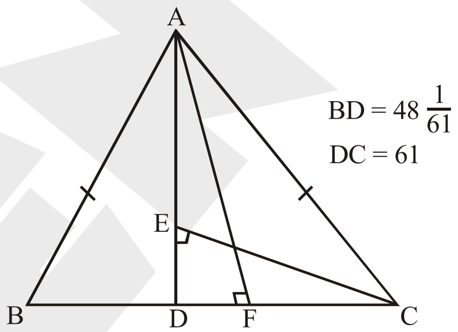
$= 2^{18} \cdot 3^{13} \cdot 5^3$

For M to be c perfect square, M should be miltiplied by $3 \cdot 5 = 15$

7. Let ABC be a triangle with AB = AC. Let D be a point on the segment BC such that $BD = 48\frac{1}{61}$ and DC = 61. Let E be a point on AD such that CE is perpendicular to AD and DE = 11. Find AE.

Ans. (25)

Sol. Right ange ΔCED



$CE = \sqrt{61^2 - 11^2}$

$= \sqrt{72 \times 50}$

$= 6 \times 10 = 60$

Hence ABC is isosceles Δ

$\Rightarrow$ AF Divide

BC = BF + FC (BF = FC)

$48 + \frac{1}{61} + 61 = 2BF$

$109 + \frac{1}{61} = 2BF$

$54 + \frac{1}{2} + \frac{1}{122} = BF$

$54 + \frac{61+1}{122} = BF$

$54 + \frac{31}{61} = BF$

In right angle ΔAFC

$AC^2 = \left(54 + \frac{31}{61}\right)^2 + AF^2 \qquad .... (1)$

In Right angle ΔAEC

$AC^2 = \left(60^2\right) + \left(AE\right)^2 \qquad .... (2)$

$DF = BF - BD = 54 + \frac{31}{61} - 48 - \frac{1}{61}$

$= 6 + \frac{30}{61}$

So in Right angle triangle AFD

$AD^2 = AF^2 + DF^2$

$(AE + ED)^2 - DF^2 = AF^2$

$\left(AE^2 + 11\right)^2 - \left(6 + \frac{30}{31}\right)^2 = AF^2$

Equation (i) and (ii)

$$\Rightarrow \quad \left(54+\frac{31}{61}\right)^2 + \left(AE+11\right)^2 = \left(6+\frac{30}{31}\right)^2 = (60)^2 + AE^2$$

On solving AE = 25

8. A 5-digit number (in base 10) has digits k, k + l, k + 2, 3k, k + 3 in that order, from left to right. If this number is m^2 for some natural number m, find the sum of the digits of m.

Ans. (15)

Sol. According to the question,

$m^2 = 10^4\,(K) + 10^3\,(K + 1) + 10^2\,(K + 2) + 10^1\,(3K) + K + 3$

$= 10^4 K + 10^3 K + 10^2 K + 10\,(3K) + K + 10^3 + 10^2 \cdot 2 + 3$

$= K\,(10^4 + 10^3 + 10^2 + 31) + 1203$

$= K(11131) + 1203$

For K = 3

$\Rightarrow \quad m^2 = 34{,}596$

$\Rightarrow \quad m = 186$

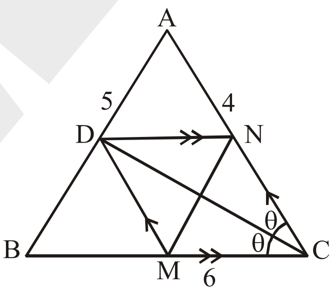
$\therefore \quad$ Sum of the digits of m = 15

9. Let ABC be a triangle with AB = 5, AC = 4, BC = 6. The internal angle bisector of c intersects the side AB at D. Points M and N are taken on sides BC and AC, respectively, such that DM || AC and DN || BC. If $(MN)^2 = \dfrac{p}{q}$ where p and q are relatively prime positive integers then what is the sum of the digits of $|p - q|$?

Ans. (02)

Sol. CD is the bisector of $\angle C$

$$\Rightarrow \quad \frac{AD}{BD} = \frac{4}{6} = \frac{2}{3}$$



$\therefore \quad AD = 2$ and $BD = 3$

Now, DH || BC

$$\therefore \quad \frac{AD}{BD} = \frac{AN}{MC} = \frac{2}{3}$$

$$\therefore \quad NC = \frac{3}{5} \times 4 = \frac{12}{5}$$

AlsoDM || AC

$$\Rightarrow \quad \frac{BM}{MC} = \frac{BD}{DA} = \frac{3}{2}$$

$$\therefore \quad MC = \frac{2}{5} \times 6 = \frac{12}{5}$$

So, DMCM is a rhombus.

Now, $\cos C = \dfrac{6^2+4^2-5^2}{2\cdot 6\cdot 4} = \dfrac{\left(\frac{12}{5}\right)^2+\left(\frac{12}{5}\right)^2-(MN)^2}{2\left(\frac{12}{5}\right)^2}$

On solving integer get

$MN^2 = \dfrac{126}{25} = \dfrac{p}{q}$

$\therefore$ $|p-q| = 101$

$\Rightarrow$ Sum of the digits of $|p-q| = 1 + 0 + 1 = 2$

10. Five students take a test on which any integer score from 0 to 100 inclusive is possible. What is the largest possible difference between the median and the mean of the scores? (The median of a set of scores is the middlemost score when the data is arranged in increasing order. It is exactly the middle score when there are an odd number of scores and it is the average of the two middle scores when there are an even number of scores.)

Ans. **(40)**

Sol. 0, 0, 0, 100, 100

$\text{difference}_{\max} = |\text{median} - \text{mean}|$

median = 0

$\text{mean} = \dfrac{0+0+0+100+100}{5} = 40$

$\text{difference}_{\max} = |40-0| = 40$

11. Let X = {– 5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5} and S = {(a, b) $\in$ X $\times$ X : $x^2 + ax + b$ and $x^3 + bx + a$ have at least a common real zero}. How many elements are there in S?

Ans. **(24)**

Sol.
$$
\begin{array}{r|l}
 & x - a \\
\hline
x^2+ax+b & x^3 + bx + a \\
 & \underline{x^3 + ax^2 + bx} \\
 & -ax^2 + a \\
 & \underline{-ax^2 - a^2x - ab} \\
 & a^2x + a(b+1)
\end{array}
$$

If a common root is there, then $a^2x + a(b+1)$ must be a factor of $x^2 + ax + b$. So, for $x = -\left(\dfrac{b+1}{a}\right)$ is a root of $x^2 + ax + b$.

$\left(\dfrac{b+1}{a}\right)^2 - a\dfrac{(b+1)^2}{a} + b = 0$

$a = \pm(b+1)$

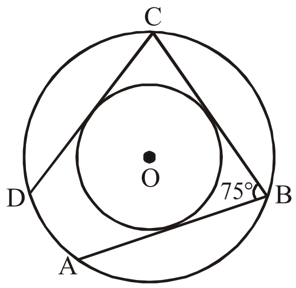
$a - b = 1$ or $a + b = -1$

S {(–1, 0), (0, –1), (–2, 1), (1, –2),(–3, 2), (2, –3),(–4, 3), (3, –4),(–5, 4), (4, –5),(2, 1), (3, 2),(4, 3), (5, 4),(–1, –2), (–2, – 3),(–3, –4), (–4, –5),(1, 0)}

(0, –2), (0, –3), (0, –4), (0, –5) (0, 6) are also the solution.

therefore total common solution = 24

12. Given a pair of concentric circles, chords AB, BC, CD, ... of the outer circle are drawn such that they all touch the inner circle. If $\angle ABC = 75°$, how many chords can be drawn before returning to the starting point?



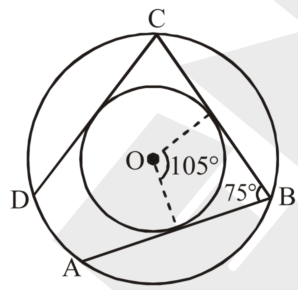
Ans. **(24)**

Sol. Here, central angle = 105°

So, total number of chords

$$= \frac{360°}{\gcd(105°, 360°)}$$

$$= \frac{360°}{15°} = 24$$



13. Find the sum of all positive integers n for which $|2^n + 5^n - 65|$ is a perfect square.

Ans. **(06)**

Sol. Let $m^2 = |2^n + 5^n - 65|$

For $n = 2$, $m^2 = |4 + 25 - 65| = |-36| = 6^2$

For $n \geq 3$

$m^2 = |2^n + 5(5^{n-1} - 13)|$

If $n = 4$, $m^2 = 576 = 24^2$

If $n \geq 5$, $m^2 = 2^n + 5\,(ab\,25 - 13)$

$= 2^n + 5 \cdot ab\,12$

$= 2^n + xab\,60$ (where $x \in I^+$)

So, for n = 1, 3, 5, 7 ..... not possible as unit digit is 2 and 8.

Also, for $n = 2k$

$m^2 = 4k + 5\,(5^{2k-1} - 13)$

$\Rightarrow m^2 - (2^k)^2 = 5\,(5^{2k-1} - 13)$

$\Rightarrow (m - 2^k)(m + 2^k) = 5\,(5^{2k-1} - 13)$

Here, last 2 digit always ends with 60 in RHS. So, not possible for n = 6, 8, 10, ...

$\therefore$ n = 2 and 4

$\Rightarrow$ Sum of n = 6

14. The product $55 \times 60 \times 65$ is written as the product of five distinct positive integers. What is the least possible value of the largest of these integers?

Ans. (20)

Sol. $55 \times 60 \times 65$

$N = 5 \times 11 \times 12 \times 5 \times 13 \times 5$

$N = 5 \times 11 \times 13 \times 15 \times 20$

So, least value of largest of these integers = 20

15. Three couples sit for a photograph in 2 rows of three people each such that no couple is sitting in the same row next to each other or in the same column one behind the other. How many arrangements are possible?

Ans. (96)

Sol. Couples be

	C_1	C_2	C_3
R_1	____	____	____
R_2	____	____	____

B_1, G_1

B_2, G_2

B_3, G_3

Case I : -

B_1, B_2, B_3 are in same row

then G_1, G_2, G_3 can be arrange in other row in

Derrangement (3) = 2 ways

So, $(\underbrace{3!}_{\text{Arrange } B_1, B_2, B_3} \times \underbrace{2)}_{\text{Arrange } G_1, G_2, G_3} \times \underbrace{2}_{R_1, R_2} = 24$ ways

Case II :-

Two boys one girl in a row

Say if B_1, B_2 then girl cannot be G_1 or G_2

because if say B_1, B_2, G_1 in a row

$$\begin{array}{cccc} & \underline{B_1} & \underline{B_2} & \underline{G_1} \\ G_2, G_3, B_3 \longrightarrow & \_\_\_\_ & \_\_\_\_ & \_\_\_\_ \end{array}$$

Not possible

B_1, B_2, G_3 in a row - 3!

In other row B_3, G_1, G_2

in derrangement (3) = 2 ways

$(3! \times 2) \times 2 = 24$

Similarly if $B_1, B_3, G_2 = 24$ ways

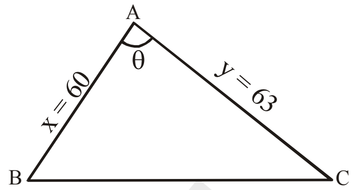
$B_2, B_3, G_1 = 24$ ways

Total = 96 ways

16. The sides r and y of a scalene triangle satisfy $x+\frac{2\Delta}{x}=y+\frac{2\Delta}{y}$, where Δ is the area of the triangle. If $x = 60$, $y = 63$, what is the length of the largest side of the triangle?

Ans. (87)

Sol. $x + \frac{2\Delta}{x} = y + \frac{2\Delta}{y}$



$\Rightarrow \quad x + 2 \cdot \frac{1}{2} xy \frac{\sin\theta}{x} = y + 2\frac{1}{2} xy \frac{\sin\theta}{y}$

$\Rightarrow \quad x + y \sin\theta = y + x \sin \theta$

$\Rightarrow \quad x - y = \sin \theta \, (x - y)$

$\Rightarrow \quad \sin \theta = 1$

$\Rightarrow \quad \theta = 90$

$\therefore \quad BC = 87$

$\Rightarrow$ Largest side of $\Delta ABC = 87$

17. How many two digit numbers have exactly 4 positive factors? (Here 1 and the number n are also considered as factors of n.)

Ans. (30)

Sol. If P_1 & P_2 are prime then

$N = P_1 \times P_2$ or $N = P^3$.

Primes less than so are

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

for $P_1 = 2 \quad P_2 \in (5, ... 47) = 13$

$P_1 = 3 \quad P_2 \in (5, ... 31) = 9$

$P_1 = 5 \quad P_2 \in (7, 11, ... 19) = 5$

$P_1 = 7 \quad P_2 \in (71, 13, 1) = 2$

$P^3 = 3^3$

Total number = 13 + 9 + 5 + 2 + 1 = 30

18. If

$$\sum_{k=1}^{40}\left(\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}\right)=a+\frac{b}{c}$$

where a, b, c $\in \mathbb{N}$, b < c,gcd(b, c) = 1, then what is the value of a +b?

Ans. (80)

Sol. $\sum_{k=1}^{40}\left(\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}\right)=a+\frac{b}{c}$

$$\Rightarrow \sum_{k=1}^{40}\left(\sqrt{1+\frac{k^2+1+2k+k^2}{k^2(k+1)^2}}\right)=a+\frac{b}{c}$$

$$\Rightarrow \sum_{k=1}^{40}\left(\sqrt{(1)^2+\frac{2k^2+2k}{k^2(k+1)^2}+\frac{1}{k^2(k+1)^2}}\right)=a+\frac{b}{c}$$

$$\Rightarrow \sum_{k=1}^{40}\sqrt{(1)^2+\frac{2}{k(k+1)}+\left(\frac{1}{k(k+1)}\right)^2}=a+\frac{b}{c}$$

$$\Rightarrow \sum_{k=1}^{40}1+\frac{1}{k(k+1)}=a+\frac{b}{c}$$

$$\Rightarrow \sum_{k=1}^{40}1+\sum_{k=1}^{40}\frac{1}{k(k+1)}=a+\frac{b}{c}$$

$$= 40 + \sum_{k=1}^{40}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

$$\Rightarrow 40+\left(1-\frac{1}{41}\right)=a+\frac{b}{c}$$

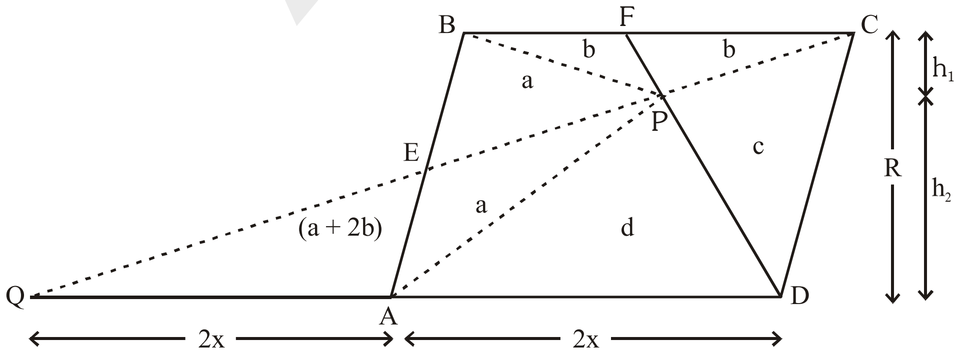
$$\Rightarrow 40+\frac{40}{41}=a+\frac{b}{c}$$

a + b = 40 + 40 = 80

19. Let ABCD be a parallelogram. Let E and F be midpoints of AB and BC respectively. The lines EC and FD intersect in P and form four triangles APB, BPC, CPD and DPA. If the area of the parallelogram is 100 sq. units, what is the maximum area in sq. units of a triangle among these four triangles?

Ans. (40)

Sol.



$a + b = 25$

$b + c = 25$

$2a + 2b + c + d = 100$

$\Delta EQA \cong \Delta EBC$

$\Rightarrow$ $[ECQ] = a + 2b$

$\Delta FPC \sim \Delta PDQ$

$\Rightarrow$ Area of $\Delta FCP = \frac{h_1 . x}{2}$

$$\frac{h_1}{h_2} = \frac{x}{4x} = \frac{1}{4}$$

$\Rightarrow$ $h_1 = \frac{1}{5}h$

$\Rightarrow$ $[FCP] = \frac{hx}{10} = \frac{2hx}{20} = \frac{100}{20} = 5$

$\Rightarrow$ $a = 15$

$b = 5$

$c = 20$

$d = 100 - 2a - 2b - c = 40$

20. A group of women working together at the same rate can build a wall in 45 hours. When the work started, all the women did not start working together. They joined the work over a period of time, one by one, at equal intervals. Once at work, each one stayed till the work was complete. If the first woman worked 5 times as many hours as the last woman, for how many hours did the first woman work?

Ans. (75)

Sol. Let there are 'n' women

$\Rightarrow$ Each woman's one hour work $= \frac{1}{45n}$

Also, $5\left[t - (n-1)d\right] = t$

$\Rightarrow$ $4t = 5(n - 1)d$

$\Rightarrow$ $\frac{1}{45n}\left(\frac{n}{2}\right)\left[2t - (n-1)d\right] = 1$

$\Rightarrow$ $\frac{1}{90}\left[2t - \frac{4t}{5}\right] = 1$

$\Rightarrow$ $t = 75$ hours

21. A total fixed amount of N thousand rupees is given to three persons A, B, C, every year, each being given an amount proportional to her age. In the first year, A got half the total amount. When the sixth payment was made, A got six-seventh of the amount that she had in the first year; B got Rs 1000 less than that she had in the first year; and C got twice of that she had in the first year. Find N.

Ans. (35)

Sol.

	A	B	C
Age at begining	a	b	c
Money at first year	$\frac{N}{2}$	$\frac{b}{b+c}\left(\frac{N}{2}\right)$	$\frac{c}{b+c}\left(\frac{N}{2}\right)$
Age at 6^{th} payment	a + 5	b + 5	c + 5
Money recieved	$\frac{6}{7}\left(\frac{N}{2}\right)=\frac{3N}{7}$	$\frac{b}{b+c}\left(\frac{N}{2}\right)-1000$	$\frac{c}{b+c}(N)$

Amount α age $\Rightarrow$ a = b + c

At 6^{th} payment, $\frac{a+5}{a+b+c+15}=\frac{3}{7}$

$$\Rightarrow \frac{b+c+5}{2b+2c+15}=\frac{3}{7} \Rightarrow \boxed{b+c=10}$$

For C, at 6^{th} payment : $\frac{c+5}{b+c+10}\times\frac{4N}{7}=\frac{c}{b+c}\times N$

$$\frac{4}{7}\left(\frac{c+5}{20}\right)=\frac{c}{10}$$

$\Rightarrow c = 2 \Rightarrow b = 8$

For B, at 6^{th} payment :

$$\frac{b+5}{b+c+10}\times\frac{4N}{7}=\frac{b}{b+c}\left(\frac{N}{2}\right)-1000$$

$$\Rightarrow \frac{13}{20}\times\frac{4N}{7}=\frac{8}{10}\left(\frac{N}{2}\right)-1000$$

$$\Rightarrow N\left(\frac{2}{5}-\frac{13}{35}\right)=1000$$

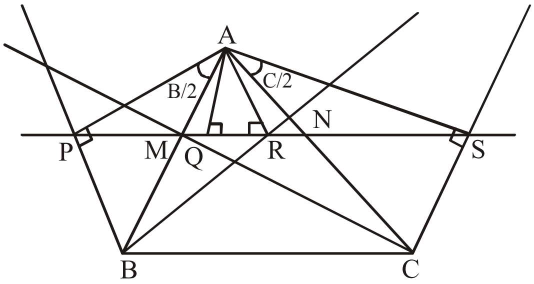
$\Rightarrow$ N = 35,000

N = 35

22. In triangle ABC, let P and R be the feet of the perpendiculars from A onto the external and internal bisectors of $\angle ABC$, respectively; and let Q and S be the feet of the perpendiculars from A onto the internal and external bisectors of $\angle ACB$, respectively. If PQ = 7, QR = 6 and RS = 8, what is the area of triangle ABC?

Ans. **(84)**

Sol.



Let M, N be the midpoints of AB, AC. Join PM, NS

M is circumcentre of right angled

$\Delta APB \Rightarrow \angle MAP = \angle MPA = B/2$

$\Rightarrow \angle BMP = B = \angle MBC$

$\Rightarrow PM \parallel BC$

Similarly $NS \parallel BC$

$MN \parallel BC$ (midpoint theorem)

$\Rightarrow$ P, M, N, S are collinear

Now, Join MR, NQ

M is circumcentre of right angled ΔARB

$MR = MB \Rightarrow \angle MRB = \angle MBR = B/2 = \angle RBC$

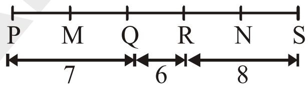
$\Rightarrow MR \parallel BC$

Similarly $NQ \parallel BC$

M, Q, R, N are collinear

$\Rightarrow$ PMQRNS is a straight line.

Also PR, QS are diameterrs of cyclic quadrilateral APBR, ASCQ = PM = MR and QN = NS



$$PM = MR = \frac{13}{2},\ QN = NS = \frac{14}{2} = 7$$

$$\Rightarrow PA = PB = \frac{13}{2},\ NA = NC = 7$$

$$\Rightarrow AB = 13 \Rightarrow AC = 14$$

$$MN = MR + QN - QR = \frac{13}{2} + 7 - 6 = \frac{15}{2}$$

$\Rightarrow$ BC = 2 MN = 15 (Mid point Theorem)

Sides are 13, 14, 15

$\Rightarrow \therefore$ area = 84 (Heron's formula)

23. The incircle of a scalene triangle ABC touches BC at D, CA at E and AB at F. Let r_A be the radius of the circle inside ABC which is tangent to Γ and the sides AB and AC. Define r_B and r_C similarly. If $r_A = 16$, $r_B = 25$ and $r_C = 36$, determine the radius of Γ.

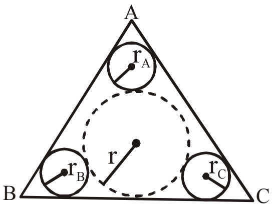
Ans. **(74)**

Sol. Using the formula

$$r = \sqrt{r_a \cdot r_b} + \sqrt{r_b \cdot r_c} + \sqrt{r_c \cdot r_a}$$

$$= \sqrt{16 \cdot 25} + \sqrt{25 \cdot 36} + \sqrt{36 \cdot 16}$$

$$= 20 + 30 + 24 = 74$$



24. A light source at the point (0, 16) in the coordinate plane casts light in all directions. A disc (a circle along with its interior) of radius 2 with center at (6, 10) casts a shadow on the X axis. The length of the shadow can be written in the form $m\sqrt{n}$ where m, n are positive integers and n is square-free. Find m + n.

Ans. **(21)**

Sol. Here, PQ is the required length of shadow.

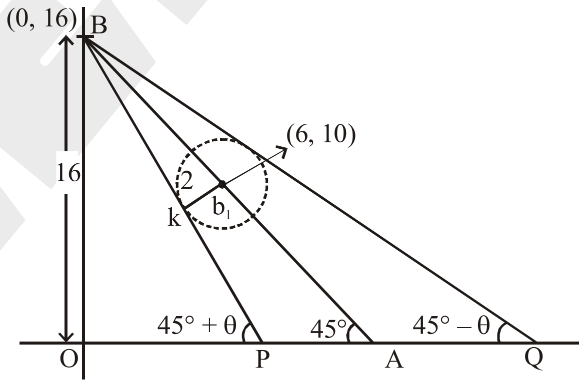
Now, slope of $BO_1 = -1$

Let $\angle PBA = \angle ABQ = \theta$

Then,

$\angle OPB = 45° + \theta$, $\qquad \angle BQO = 45° - \theta$

Also, $OB = 16$, $O_1B = 6\sqrt{2}$



$\therefore \quad OP = \dfrac{16}{\tan(45+\theta)}$ and $OQ = \dfrac{16}{\tan(45-\theta)}$

So, $PQ = OQ - OP$

$$= 16\left(\frac{1}{\tan(45-\theta)} - \frac{1}{\tan(45+\theta)}\right)$$

$$= 16\left(\frac{\tan\theta + 1}{-\tan\theta + 1} - \frac{-\tan\theta + 1}{\tan\theta + 1}\right)$$

$$= 16\left(\frac{4\tan\theta}{-\tan^2\theta + 1}\right) \qquad \text{.... (1)}$$

But, In ΔBO_1K, $\tan\theta = \dfrac{2}{2\sqrt{17}} = \dfrac{1}{\sqrt{17}}$

$$\therefore \quad PQ = \frac{16.4\left(\frac{1}{\sqrt{17}}\right)}{1 - \left(\frac{1}{\sqrt{17}}\right)^2}$$

$$PQ = 4\sqrt{17} = m\sqrt{n}$$

So, $\boxed{m + n = 21}$

25. For a positive integer n, let $\langle n \rangle$ denote the perfect square integer closest to n. For example, $\langle 74 \rangle$ = 81, $\langle 18 \rangle$ = 16. If N is the smallest positive integer such that

$$\langle 91 \rangle \cdot \langle 120 \rangle \cdot \langle 143 \rangle \cdot \langle 180 \rangle \cdot \langle N \rangle = 91 \cdot 120 \cdot 143 \cdot 180 \cdot N$$

find the sum of the squares of the digits of N.

Ans. (56)

Sol. $\langle 91 \rangle = 100$

$\langle 120 \rangle = 121$

$\langle 143 \rangle = 144$

$\langle 180 \rangle = 169$

$\therefore \quad 81 \cdot 121 \cdot 144 \cdot 169 \cdot \langle N \rangle = 91 \cdot 120 \cdot 143 \cdot 180 \cdot N$

$\Rightarrow \quad \langle N \rangle = \dfrac{91 \cdot 120 \cdot 143 \cdot 180 \cdot N}{100 \cdot 121 \cdot 144 \cdot 169}$

$\Rightarrow \quad \langle N \rangle = \dfrac{21}{22} N$

Now to make $\langle N \rangle$ to be a perfect square, we can take smallest N to be $2 \cdot 11 \cdot 3 \cdot 7 = 162$

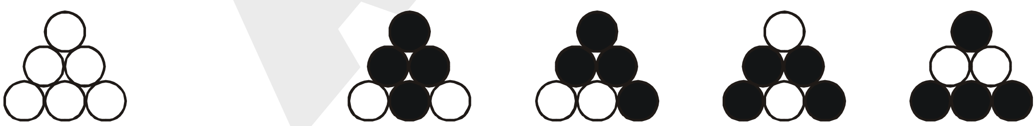
$\therefore \quad \langle N \rangle = \dfrac{21}{22} N = \dfrac{3 \cdot 7 \cdot 2 \cdot 11 \cdot 3 \cdot 7}{2 \cdot 11} = (21)^2 = 441$

Which is the nearest perfect square to 462.

$\therefore$ Sum of square of digits of 462 is $4^2 + 6^2 + 2^2$

$= 16 + 36 + 4 = 56$

26. In the figure below, 4 of the 6 disks are to be colored black and 2 are to be colored white. Two colorings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same.



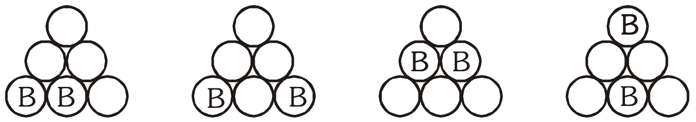
(i) (ii) (iii) (iv)

Fig.1

There are only four such colorings for the given two colors, as shown in Figure 1. In how many ways can we color the 6 disks such that 2 are colored black, 2 are colored white, 2 are colored blue with the given identification condition?

Ans. (18)

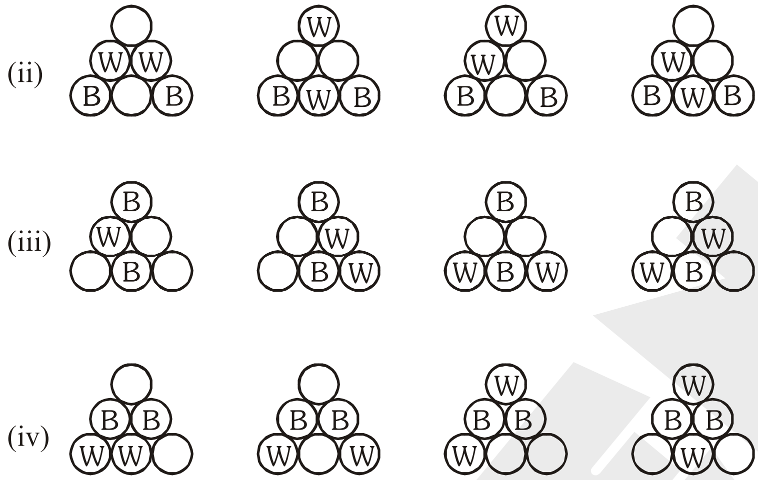
Sol. We can colour 2 black in following four mutually exclusive ways



(i) (ii) (iii) (iv)

In (i) 2 W and 2 Blue can be placed in $\frac{4!}{2!2!} = 6$ ways

In (ii), (iii), (iv) we have 4 ways to colour as shown :

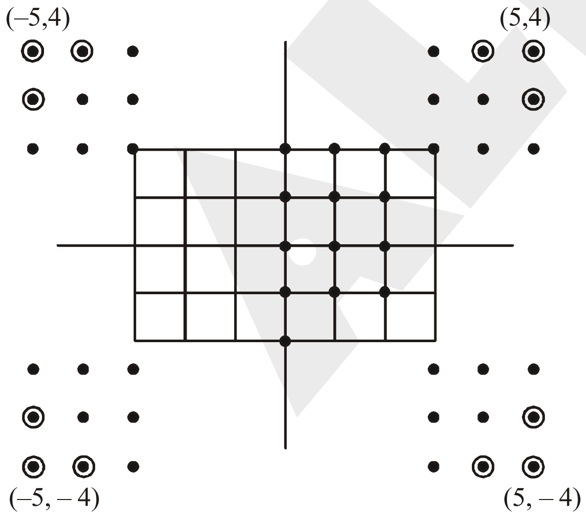


Hence total ways $= 6 + 3 \times 4 = 18$ ways.

27. A bug travels in the coordinate plane moving only along the lines that are parallel to the x axis or y axis. Let A = (–3, 2) and B(3, –2). Consider all possible paths of the bug from A to B of length at most 14. How many points with integer coordinatess lie on at least one of these paths?

Ans. (87)

Sol.



So, there are total of 11 vertical line and 9 horizontal lines creating a total of 99 integral coordinates.

Now, among these 99 points, the points which are marked in red, these must se excluded, as bug has to coverup at most 14 steps (i.e., length = 14)

So, there are 99 – 12 = 87 points with integer coordinates.

28. A natural number n is said to be good if n is the sum of r consecutive positive integers, for some $r \geq 2$. Find the number of good numbers in the set {1, 2, ..., 100}.

Ans. **(93)**

Sol. Let us check several values of r starting from 2 i.e.

$r = 2 : \{(1, 2); (2, 3); (3, 4); .... (49, 50)\}$

These are 49 pairs with n = 3, 5, 7 99

$r = 3 : \{(1, 2, 3); (2, 3, 4); ....(32, 33, 34)\}$

These are 32 pairs with n = 6, 9, 12, ... 99

$r = 4 : \{(1, 2, 3, 4); (2, 3, 4, 5); (3, 4, 5, 6) .... (23, 24, 25, 26)\}$

These are 23 pairs with n = 10, 14, ... 98

$r = 5 : \{(1, 2, 3, 4, 5) : .... (18, 19, 20, 21, 22)\}$

These are 18 pairs with n = 15, 20, ... 100.

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$r = 11 : \{(1, 2, ...11); (2, 3, ...12); .... (4, 5 ....14)\}$

$r = 12 : \{(1, 2, ...12); (2, 3, ...13)\}$

$r = 13 : \{(1, 2, ...13)\}$

observing this, we observe that n = 1, 2, 4, 8, 16, 32, 64 is not coming.

$\therefore$ Good numbers = 100 – 7

= 93

29. Positive integers a, b, c satisfy $\frac{ab}{a-b} = c$. What is the largest possible value of a + b + c not exceeding 99?

Ans. **(99)**

Sol. $a, b, c \in I^{+}$

$$\frac{ab}{a-b} = c$$

$$\frac{a-b}{ab} = \frac{1}{c}$$

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c}$$

$$\frac{1}{b} = \frac{1}{a} + \frac{1}{c}$$

Let gcd (a, c) = k, then then a = kx, c = ky, which in turns gives b also to be a multiple of k. So, a,b,c all must be a multiple of k.

One possible solution is $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$

i.e., a = 3, b = 2, c = 6

So, other solutions can be taken

a = 3k, b = 2k, c = 6k

Now, $a + b + c \leq 99$

$kx + ky + kz \leq 99$

$k(x + y + z) \leq 99 = 9 \times 11$

Lets check for the largest possible value i.e, 99

$k(x + y + z) = 99$

Suppose k = 9 and considering

$a = 3 \times 9 = 27$

$b = 2 \times 9 = 189$

and $c = 6 \times 9 = 54$, and checking $\frac{1}{18} = \frac{1}{27} + \frac{1}{54}$ which is satisfying and giving the largest possible value of a + b + c to be 18 + 27 + 54 = 99

30. Find the number of pairs (a, b) of natural numbers such that b is a 3-digit number, a +1 divides b – 1 and b divides $a^2 + a + 2$.

Ans. **(16)**

Sol. $a, b \in N$

$a + 1 \mid b - 1$

$b \mid a^2 + a + 2$

let $b - 1 = k(a + 1)$, where k is any positive integer.

$b = ka + k + 1$

Now $ka + k + 1 \mid a^2 + a + 2$

$\Rightarrow \quad ka + k + 1 \mid ka^2 + ka + 2k$

$ka + k + 1 \mid ka^2 + ka + 2k - (ka^2 + ka + a)$

$ka + k + 1 \mid 2k - a$

Put

$a = 2x$, such that $ka + k + 1 \mid 0$

then $b = k(2k) + k + 1$

$= 2k^2 + k + 1$

Now, $100 \leq b \leq 999$

So, $100 \leq 2k^2 + 1 \mid \leq 999$

Checking certain values of k, we get $k \in [7, 22]$

these are 16 possible values by b.

$\therefore$ 16 pairs of (a, b).

Alternate Solution :

a, b $\in$ N and b is 3 digit

$a + 1 \mid b - 1 \quad \Rightarrow \quad b - 1 = (a + 1)k$ for some $k \in N$

$b \mid a^2 + a + 2 \Rightarrow a^2 + a + 2 = b\ell$ for some $\ell \in N$

$\Rightarrow \quad a^2 + a + 2 = [(a + 1)k + 1]\ell$

$\Rightarrow \quad a^2 + a(1 - k\ell) + 2 - k\ell - \ell = 0$

$D = (k\ell - 1)^2 - 4(2 - k\ell - \ell)$

$D = (k\ell + 1)^2 + 4(\ell - 2)$

$\because$ a $\in$ N, D = perfect square = λ^2 for some $\lambda \in N$

$(k\ell + 1)^2 + 4(\ell - 2) = \lambda^2$

Case 1 : If $\ell = 2$, $D = (k\ell + 1)^2$

$$a = \frac{k\ell - 1 + k\ell + 1}{2} = k\ell = 2k$$

$$b = (a + 1)\,k + 1 = 2k^2 + k + 1$$

k = 7, b = 106, a = 14

k = 8, b = 137, a = 16

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k = 22, b = 991, a = 44

16 pairs of (a, b)

Case 2 : If $\ell \neq 2 \Rightarrow \lambda^2 - (k\ell + 1) = 4(\ell - 2)$

$\Rightarrow \quad (\lambda + k\ell + 1)(\lambda - k\ell - 1) = 4(\ell - 2)$

$\Rightarrow$ Both $\lambda + k\ell + 1$ and $\lambda - k\ell - 1$ must be even

$\lambda + k\ell + 1 = 2(\ell - 2)$	$\lambda + k\ell + 1 = \ell - 2$
$\lambda - k\ell$	$\lambda - k\ell - 1 = 4$
$2(k\ell + 1) = 2(\ell - 3)$	$2(k\ell + 1) = \ell - 6$
$k\ell = \ell - 4$	$k\ell + 1 = \frac{\ell}{2} - 4$
$>\ell \quad <\ell$	$>\ell \quad <\ell$
Not possible	Not possible

If we further reduce the value of $\lambda + k\ell + 1$ and increase $\lambda - k\ell - 1$ we can see that after subtracting both equation, LHS $> \ell$ but RHS $< \ell$.

Hence only Case 1 ($\ell = 2$) is possible & 16 ordered pairs.