

Date: 06/03/2022

Max. Marks: 40

Time : 75 Minutes

**(PART-A)
INSTRUCTIONS**

1. Use of mobile phones, smart phones, ipads, calculators, programmable wrist watches is **STRICTLY PROHIBITED**. Only ordinary pens and pencils are allowed inside the examination hall.
2. The correction is done by machines through scanning, On the OMR Sheet, darken bubbles completely with a black or blue ball pen Please **DO NOT** use a pencil or a gel pen. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.
3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your score.
4. Incompletely, Incorrectly or carelessly filled Information may disqualify your candidature.
5. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.
6. Questions 1 and 2 carry 2 marks each; questions 3 to 9 carry 3 marks each; questions 10 to 12 carry 5 marks each
7. All questions are compulsory,
8. There are no negative marks.
9. Do all rough work In the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.
10. After the exam, you may take away the Candidate's copy of the OMR sheet.
11. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.
12. You may take away the question paper after the examination.

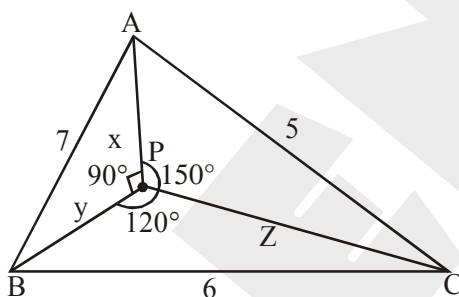
But ; $\angle ADX = \angle ZDX$
 $\Rightarrow \angle AXD = \angle ADX$
 $\Rightarrow AD = AX$
 $\Rightarrow \square AXZD$ will be rhombus
 $\Rightarrow \text{ar}(\square AXZD) = 20$
 $\Rightarrow \text{ar}(\square ABCD) = 40$

6. Let x, y, z be positive real numbers such that $x^2 + y^2 = 49$, $y^2 + yz + z^2 = 36$ and $x^2 + \sqrt{3}xz + z^2 = 25$.

If the value of $2xy + \sqrt{3}yz + zx$ can be written as $p\sqrt{q}$ where p, q are integers and q is not divisible by square of any prime number, find $p + q$.

Ans. 30

Sol. Consider $\triangle ABC$ such that $AB = 7$, $BC = 6$, $AC = 5$ such that $\angle APB = 90^\circ$, $\angle BPC = 120^\circ$ & $\angle APC = 150^\circ$ then using cosine rule in $\triangle APB$, $\triangle BPC$ & $\triangle CPA$ gives $x^2 + y^2 = 49$, $y^2 + yz + z^2 = 36$ and $x^2 + \sqrt{3}xz + z^2 = 25$ respectively.



Now

$$[ABC] = [APB] + [BPC] + [APC]$$

$$\sqrt{9(9-5)(9-6)(9-7)} = \frac{1}{2} \cdot xy + \frac{1}{2} xz \cdot \sin 120^\circ + \frac{1}{2} xz \sin 150^\circ$$

$$6\sqrt{6} = \frac{1}{2} \left(xy + \frac{\sqrt{3}}{2} yz + \frac{1}{2} xz \right)$$

$$\Rightarrow 12\sqrt{6} = \frac{1}{2} (2xy + \sqrt{3}yz + xz)$$

$$\Rightarrow 24\sqrt{6} = 2xy + \sqrt{3}yz + zx$$

$$\therefore p + q = 24 + 6 = 30$$

7. Find the number of maps $f : (1, 2, 3) \rightarrow \{1, 2, 3, 4, 5\}$ such that $f(i) \leq f(j)$ whenever $i < j$.

Ans. 35

Sol. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$

$$f(i) \leq f(j) \quad \forall i < j$$

Let $i = 1$

$$f(1) \leq f(2) \leq f(3)$$

Case-I

$$f(1) = 1, \quad f(2) = 1 \Rightarrow f(3) \geq 1$$

$$f(3) = \{1, 2, 3, 4, 5\} \quad (5)$$

$$f(2) = 2 \Rightarrow f(3) \geq 2$$

$$f(3) = \{2, 3, 4, 5\} \quad (4)$$

$$f(2) = 3 \Rightarrow f(3) \geq 3$$

$$f(3) = \{3, 4, 5\} \quad (3)$$

$$f(2) = 4, \quad f(3) \geq 4$$

$$f(3) = \{4, 5\} \quad (2)$$

$$f(2) = 5 \Rightarrow f(3) \geq 5$$

$$f(3) = \{5\} \quad (1)$$

$$\text{Total cases for } f(1) = 1 \Rightarrow (5 + 4 + 3 + 2 + 1) = 15$$

$$f(1) = 2 \Rightarrow \{4 + 3 + 2 + 1\} = 10$$

$$f(1) = 3 \Rightarrow \{3 + 2 + 1\} = 6$$

$$F(1) = 4 \Rightarrow \{2 + 1\} = 3$$

$$F(1) = 5 \Rightarrow \{1\} = 1$$

$$\text{Total cases} = 15 + 10 + 6 + 3 + 1 = 35$$

Alternate solutions

$$\text{Total cases} = \sum_{r=1}^5 \sum_{j=r}^5 (6-j)$$

$$= \sum_{r=1}^5 \{(6-r) + (6-(r+1)) + \dots + (6-5)\}$$

$$= \sum_{r=1}^5 \frac{(6-r)(7-r)}{2}$$

$$= \frac{1}{2} \sum_{r=1}^5 [42 - 13r + r^2]$$

$$= \frac{1}{2} \left[42 \times 5 - \frac{13 \times 5 \times 6}{2} + \frac{(5)(6)(11)}{6} \right]$$

$$= \frac{1}{2} [210 - 195 + 55]$$

$$= \frac{1}{2} \times 70 = 35$$

8. For any real number t , let $[t]$ denote the largest integer $\leq t$. Suppose that N is the greatest integer such that

$$\left[\sqrt{\sqrt{\sqrt{[N]}}} \right] = 4$$

Find the sum of digits of N .

Ans. 24

Sol. $\left[\sqrt{\sqrt{\sqrt{[N]}}} \right] = 4$

$$\sqrt{\sqrt{\sqrt{[N]}}} \in [4, 5)$$

$$\sqrt{\sqrt{[N]}} \in [16, 25)$$

$$[\sqrt{N}] \in [256, 625)$$

$$\sqrt{N} \in [256, 625)$$

$$N \in [65536, 390625)$$

$$N \text{ largest} = 390624$$

$$\text{Sum of digits} = 24$$

9. Let $P_0 = (3, 1)$ and define $P_{n+1} = (x_n, y_n)$ for $n \geq 0$ by $x_{n+1} = -\frac{3x_n - y_n}{2}$, $y_{n+1} = -\frac{x_n + y_n}{2}$. Find the area of the quadrilateral formed by the points $P_{96}, P_{97}, P_{98}, P_{99}$.

Ans. Bonus

Sol. $P_0 = (3, 1)$

In the question, $P_{n+1} = (x_n, y_n)$ is defined so according to it,

$P_0 = P_{-1} + 1 = (x_{-1}, y_{-1})$ which makes $n < 0$ which is not possible as $n \geq 0$ is given so it has misprint in it.

10. Suppose that P is the polynomial of least degree with Integer coefficients such that $P(\sqrt{7} + \sqrt{5}) = 2(\sqrt{7} - \sqrt{5})$. Find $P(2)$.

Ans. 40

Sol. $P(\sqrt{7} + \sqrt{5}) = 2(\sqrt{7} - \sqrt{5})$

$$P(\sqrt{7} + \sqrt{5}) = (\sqrt{7} + \sqrt{5})(\sqrt{7} - \sqrt{5})^2$$

$$\text{Let } (\sqrt{7} + \sqrt{5}) = t$$

$$P(t) = t(\sqrt{7} + \sqrt{5})^2 = t[24 - t^2]$$

$$\therefore (\sqrt{7} + \sqrt{5})^2 + (\sqrt{7} - \sqrt{5})^2 = 24$$

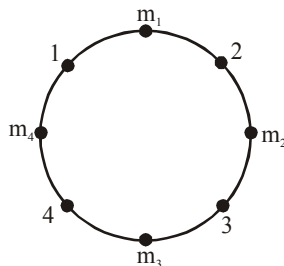
$$P(t) = 24t - t^3$$

$$P(2) = 24 \times 2 - 2^3 = 48 - 8 = 40$$

- 11.** In how many ways can four married couples sit in a merry-go-round with identical seats such that men and women occupy alternate seats and no husband sits next to his wife?

Ans. 12

Sol. Let us first sit the men. They can occupy either the odd numbered seats or the even numbered seats. Hence they can be seated in $2(4!)$ ways.



Let S_n be the number of arrangements of the women that satisfy none of the following conditions.

1. w_1 is in seat 1
2. w_1 is in seat 2
3. w_2 is in seat 2
4. w_2 is in seat 3
5. w_3 is in seat 3
6. w_3 is in seat 4
7. w_4 is in seat 4
8. w_4 is in seat 1

Then the required number of couple arrangements is $2(4!)S_n$.

We now find S_n . The 4 women can be seated in 4 seats in $4!$ ways. Let A_i be the set of arrangements that satisfy the i_{th} conditions. By Principle of inclusion exclusion, we have.

$$S_n = 4! - |A_1 \cup A_2 \cup \dots \cup A_8|$$

$$= 4! - \sum_{i=1}^8 |A_i| + \sum_{i>j} |A_i \cap A_j| - \dots + (-1)^8 |A_1 \cap A_2 \cap \dots \cap A_8| \quad \dots(1)$$

This expression can be reduced considerably. For instance, the cardinality of the set $A_1 \cap A_2$ is zero since w_1 cannot be in both the first and the second seat. In general if any two consecutive conditions are chosen the cardinality of the corresponding set will be zero. The cardinality of the set will be non zero only if no two adjacent conditions are chosen. The number of ways to pick 2 non adjacent conditions out of the 8 conditions is

$$\frac{8}{8-2} \binom{8-2}{2}.$$

Since choosing two of the conditions fixes the places for two of the women and the other $n - 2$ women can be seated in $(8 - 2)!$ ways, it follows that the number of ways in which two conditions are satisfied is

$$\frac{8}{8-2} \binom{8-2}{2} (4 - 2)!$$

Similarly, when we select three conditions that are pairwise non adjacent, the number of ways of seating the couples is

$$\frac{8}{8-3} \binom{8-3}{3} (4 - 3)!$$

We can extend this necessarily to i conditions. Hence substituting in (1).

$$S_n = 4! - \frac{8}{8-1} \binom{8-1}{1} (4-1)! + \frac{8}{8-2} \binom{8-2}{2} (4-2)! \dots\dots\dots$$

Hence number of arrangements is

$$2(4!) \sum_{i=0}^4 (-1)^i \frac{8}{8-i} \binom{8-i}{i} (4-i)!$$

However, this counts the circular rotations of the arrangements as different. We need to divide the above count by $2n$ to get the arrangements of to rotation and is given by

$$(4-1)! \sum_{i=0}^4 (-1)^i \frac{8}{8-i} \binom{8-i}{i} (4-i)!$$

$$3! \left[\frac{8}{8} \times {}^8C_0 \times 4! - \frac{8}{7} \times {}^7C_1 \times 3! + \frac{8}{6} \times {}^6C_2 \times 2! - \frac{8}{5} \times {}^5C_3 \times 1! + \frac{8}{4} \times {}^4C_4 \times 0! \right]$$

$$= 6 [24 - 48 + 40 - 16 + 2] = 6 \times 2 = 12$$

- 12.** A 12×12 board is divided into 144 unit squares by drawing lines parallel to the sides. Two rooks placed on two unit squares are said to be non attacking If they are not In the same column or same row. Find the least number N such that If N rooks are placed on the unit squares, one rook per square, we can always find 7 rooks such that no two are attacking each other.

Ans. 73

Sol. We will generalise this answer for any n . Lets replace 12 with n and 7 with $k + 1$.

First we place nk rooks so that no $nk + 1$ rooks attack each other.

Let us first pick any k rows and fill that fully. In that case we have filled nk rooks in total. But if we pick any $k + 1$ rooks then some two must be in the same column (By pigeon hole principle), which means those two will attack each other. This creates our required construction. So if we just show that given any $nk + 1$ rooks placed, we can find some $k + 1$ which do not attack each other.

So in our case $n = 12$ and $k = 6$, hence $nk + 1 = 73$ will be our answer.