SEQUENCE & PROGRESSION

1. If sum of the first 21 terms of the series $\log_{9^{1/2}} x + \log_{9^{1/3}} x + \log_{9^{1/4}} x + ...$, where $x > 0$ is 504, then x is equal to

(1) 243 (2) 9 (3) 7 (4) 81

2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_1 = 1$, $a_2 = 1$ and $a_{n+2} = 2a_{n+1} + a_n$ for all $n \geq 1$. Then the value of $47\sum_{n=1}^{\infty}\frac{a_n}{2^{3n}}$ is equal to ________.

3. Let S_n denote the sum of first n-terms of an arithmetic progression. If $S_{10} = 530$, $S_5 = 140$, then $S_{20} - S_6$ is equal to :

(1) 1862 (2) 1842 (3) 1852 (4) 1872

4. The sum of all the elements in the set $\{n \in \{1, 2, ..., 100\}$ | H.C.F. of n and 2040 is 1$\}$ is equal to _______.

5. Let S_n be the sum of the first n terms of an arithmetic progression. If $S_{3n} = 3S_{2n}$, then the value of $\frac{S_{4n}}{S_{2n}}$ is :

(1) 6 (2) 4 (3) 2 (4) 8

6. If the value of $\left(1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+......\text{upto }\infty\right)^{\log_{(0.25)}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+.......\text{upto }\infty\right)}$ is l, then l^2 is equal to ________.

7. If $\log_3 2, \log_3(2^x - 5), \log_3\left(2^x - \frac{7}{2}\right)$ are in an arithmetic progression, then the value of x is equal to ________.

8. Let $A = \{n \in \mathbf{N} \mid n^2 \leq n + 10{,}000\}$, $B = \{3k+1 \mid k \in \mathbf{N}\}$ and $C = \{2k \mid k \in N\}$, then the sum of all the elements of the set $A \cap (B - C)$ is equal to_______ .

9. The sum of the series

$$\frac{1}{x+1}+\frac{2}{x^2+1}+\frac{2^2}{x^4+1}+.....+\frac{2^{100}}{x^{2^{100}}+1} \text{ when } x = 2$$

is:

(1) $1+\frac{2^{101}}{4^{101}-1}$ (2) $1+\frac{2^{100}}{4^{101}-1}$

(3) $1-\frac{2^{100}}{4^{100}-1}$ (4) $1-\frac{2^{101}}{4^{101}-1}$

10. If the sum of an infinite GP a, ar, ar^2, ar^3,... is 15 and the sum of the squares of its each term is 150, then the sum of ar^2, ar^4, ar^6, ... is :

(1) $\frac{5}{2}$ (2) $\frac{1}{2}$ (3) $\frac{25}{2}$ (4) $\frac{9}{2}$

11. Let a_1, a_2,....,a_{10} be an AP with common difference –3 and b_1, b_2,...., b_{10} be a GP with common ratio 2. Let $c_k = a_k + b_k$, k = 1,2,..., 10. If $c_2 = 12$ and $c_3 = 13$, then $\sum_{k=1}^{10} c_k$ is equal to _____.

12. If for $x, y \in \mathbf{R}$, $x > 0$,
$y = \log_{10}x + \log_{10}x^{1/3} + \log_{10}x^{1/9} + ...$ upto ∞ terms and $\frac{2+4+6+....+2y}{3+6+9+....+3y} = \frac{4}{\log_{10} x}$, then the ordered pair (x, y) is equal to :

(1) $(10^6, 6)$ (2) $(10^4, 6)$

(3) $(10^2, 3)$ (4) $(10^6, 9)$

13. The sum of 10 terms of the series $\frac{3}{1^2 \times 2^2}+\frac{5}{2^2 \times 3^2}+\frac{7}{3^2 \times 4^2}+....$ is :

(1) 1 (2) $\frac{120}{121}$ (3) $\frac{99}{100}$ (4) $\frac{143}{144}$

14. Three numbers are in an increasing geometric progression with common ratio r. If the middle number is doubled, then the new numbers are in an arithmetic progression with common difference d. If the fourth term of GP is $3r^2$, then $r^2 - d$ is equal to :

(1) $7 - 7\sqrt{3}$ (2) $7 + \sqrt{3}$

(3) $7 - \sqrt{3}$ (4) $7 + 3\sqrt{3}$

15. The mean of 10 numbers 7×8, 10×10, 13×12, 16×14, …. is _________.

16. Let a_1, a_2, a_3, … be an A.P. If $\frac{a_1 + a_2 + ... + a_{10}}{a_1 + a_2 + ... + a_p} = \frac{100}{p^2}$, $p \neq 10$, then $\frac{a_{11}}{a_{10}}$ is equal to :

(1) $\frac{19}{21}$ (2) $\frac{100}{121}$ (3) $\frac{21}{19}$ (4) $\frac{121}{100}$

17. The number of 4-digit numbers which are neither multiple of 7 nor multiple of 3 is ____.

18. If $S = \frac{7}{5} + \frac{9}{5^2} + \frac{13}{5^3} + \frac{19}{5^4} + ...$, then 160 S is equal to ______.

19. Let $S_n = 1 \cdot (n-1) + 2 \cdot (n-2) + 3 \cdot (n-3) + ... + (n-1) \cdot 1$, $n \geq 4$. The sum $\sum_{n=4}^{\infty}\left(\frac{2S_n}{n!} - \frac{1}{(n-2)!}\right)$ is equal to :

(1) $\frac{e-1}{3}$ (2) $\frac{e-2}{6}$ (3) $\frac{e}{3}$ (4) $\frac{e}{6}$

20. Let a_1, a_2,..., a_{21} be an AP such that $\sum_{n=1}^{20}\frac{1}{a_n a_{n+1}} = \frac{4}{9}$. If the sum of this AP is 189, then $a_6 a_{16}$ is equal to :

(1) 57 (2) 72 (3) 48 (4) 36

21. Let a, b, c be in arithmetic progression. Let the centroid of the triangle with vertices (a, c), (2, b) and (a, b) be $\left(\frac{10}{3}, \frac{7}{3}\right)$. If α, β are the roots of the equation $ax^2 + bx + 1 = 0$, then the value of $\alpha^2 + \beta^2 - \alpha\beta$ is :

(1) $\frac{71}{256}$ (2) $\frac{69}{256}$

(3) $-\frac{69}{256}$ (4) $-\frac{71}{256}$

22. The sum of first four terms of a geometric progression (G.P.) is $\frac{65}{12}$ and the sum of their respective reciprocals is $\frac{65}{18}$. If the product of first three terms of the G.P. is 1, and the third term is α, then 2α is _______.

23. If $0 < \theta, \phi < \frac{\pi}{2}$, $x = \sum_{n=0}^{\infty}\cos^{2n}\theta$, $y = \sum_{n=0}^{\infty}\sin^{2n}\phi$ and $z = \sum_{n=0}^{\infty}\cos^{2n}\theta \cdot \sin^{2n}\phi$ then :

(1) $xy - z = (x + y)z$ (2) $xy + yz + zx = z$

(3) $xyz = 4$ (4) $xy + z = (x + y)z$

24. Let A_1, A_2, A_3, ... be squares such that for each $n \geq 1$, the length of the side of A_n equals the length of diagonal of A_{n+1}. If the length of A_1 is 12 cm, then the smallest value of n for which area of A_n is less than one, is ______.

25. The sum of the series $\sum_{n=1}^{\infty}\frac{n^2+6n+10}{(2n+1)!}$ is equal to :

(1) $\frac{41}{8}e+\frac{19}{8}e^{-1}-10$

(2) $\frac{41}{8}e-\frac{19}{8}e^{-1}-10$

(3) $\frac{41}{8}e+\frac{19}{8}e^{-1}+10$

(4) $-\frac{41}{8}e+\frac{19}{8}e^{-1}-10$

26. If the arithmetic mean and geometric mean of the p^{th} and q^{th} terms of the sequence $-16, 8, -4, 2, \ldots$ satisfy the equation $4x^2 - 9x + 5 = 0$, then p + q is equal to ______.

27. In a increasing geometric series, the sum of the second and the sixth term is $\frac{25}{2}$ and the product of the third and fifth term is 25. Then, the sum of 4^{th}, 6^{th} and 8^{th} terms is equal to

(1) 30 (2) 26 (3) 35 (4) 32

28. The sum of the infinite series $1+\frac{2}{3}+\frac{7}{3^2}+\frac{12}{3^3}+\frac{17}{3^4}+\frac{22}{3^5}+\ldots$ is equal to

(1) $\frac{13}{4}$ (2) $\frac{9}{4}$ (3) $\frac{15}{4}$ (4) $\frac{11}{4}$

29. Let $\frac{1}{16}$, a and b be in G.P. and $\frac{1}{a},\frac{1}{b},6$ be in A.P., where a, b > 0. Then 72(a + b) is equal to ________.

30. Let

$$S_n(x)=\log_{a^{1/2}}x+\log_{a^{1/3}}x+\log_{a^{1/6}}x$$
$$+\log_{a^{1/11}}x+\log_{a^{1/18}}x+\log_{a^{1/27}}x+\ldots..$$

up to n-terms, where a > 1. If $S_{24}(x) = 1093$ and $S_{12}(2x) = 265$, then value of a is equal to _______.

31. Consider an arithmetic series and a geometric series having four initial terms from the set {11, 8, 21, 16, 26, 32, 4}. If the last terms of these series are the maximum possible four digit numbers, then the number of common terms in these two series is equal to ______ .

32. If 1, $\log_{10}(4^x - 2)$ and $\log_{10}\left(4^x+\frac{18}{5}\right)$ are in arithmetic progression for a real number x, then the value of the determinant $\begin{vmatrix}2\left(x-\frac{1}{2}\right) & x-1 & x^2\\ 1 & 0 & x\\ x & 1 & 0\end{vmatrix}$ is equal to :

33. If α, β are natural numbers such that $100^{\alpha} - 199\beta = (100)(100) + (99)(101) + (98)(102) +\ldots..+ (1)(199)$, then the slope of the line passing through (α,β) and origin is :

(1) 540 (2) 550 (3) 530 (4) 510

34. $\frac{1}{3^2-1}+\frac{1}{5^2-1}+\frac{1}{7^2-1}+\ldots+\frac{1}{(201)^2-1}$ is equal to

(1) $\frac{101}{404}$ (2) $\frac{25}{101}$ (3) $\frac{101}{408}$ (4) $\frac{99}{400}$

35. Let S_1 be the sum of first 2n terms of an arithmetic progression. Let S_2 be the sum of first 4n terms of the same arithmetic progression. If $(S_2 - S_1)$ is 1000, then the sum of the first 6n terms of the arithmetic progression is equal to:

(1) 1000 (2) 7000 (3) 5000 (4) 3000

36. The term independent of x in the expansion of $\left[\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right]^{10}$, $x \neq 1$, is equal to _______.

SOLUTION

1. s = 2log$_9$ x + 3 log$_9$ x+ ...... + 22 log$_9$ x

s = log$_9$ x (2 + 3 + ... + 22)

$s = \log_9 x \left\{\frac{21}{2}(2+22)\right\}$

Given $252\log_9 x = 504$

$\Rightarrow \log_9 x = 2 \Rightarrow x = 81$

2. $a_{n+2} = 2a_{n+1} + a_n$, let $\sum_{n=1}^{\infty}\frac{a_n}{8^n} = P$

Divide by 8^n we get

$$\frac{a_{n+2}}{8^n} = \frac{2a_{n+1}}{8^n} + \frac{a_n}{8^n}$$

$$\Rightarrow 64\frac{a_{n+2}}{8^{n+2}} = \frac{16a_{n+1}}{8^{n+1}} + \frac{a_n}{8^n}$$

$$64\sum_{n=1}^{\infty}\frac{a_{n+2}}{8^{n+2}} = 16\sum_{n=1}^{\infty}\frac{a_{n+1}}{8^{n+1}} + \sum_{n=1}^{\infty}\frac{a_n}{8^n}$$

$$64\left(P - \frac{a_1}{8} - \frac{a_2}{8^2}\right) = 16\left(P - \frac{a_1}{8}\right) + P$$

$$\Rightarrow 64\left(P - \frac{1}{8} - \frac{1}{64}\right) = 16\left(P - \frac{1}{8}\right) + P$$

64P – 8 – 1 = 16 P – 2 + P

47P = 7

3. $S_{10} = 530 \Rightarrow \frac{10}{2}\{2a+9d\} = 530$

$\Rightarrow$ 2a + 9d = 106 ….(1)

and S$_5$ = 140 $\Rightarrow \frac{5}{2}\{2a+4d\} = 140$

$\Rightarrow$ 2a + 4d = 56 …..(2)

$\Rightarrow$ 5d = 50 $\Rightarrow$ $\boxed{d=10}$ $\Rightarrow$ $\boxed{a=8}$

Now, $S_{20} - S_6 = \frac{20}{2}\{2a+19d\} - \frac{6}{2}\{2a+5d\}$

= 14a + 175d

= (14 × 8) + (175 × 10)

= 1862

4. 2040 = 2^3 × 3 × 5 × 17

n should not be multiple of 2, 3, 5 and 17.

Sum of all n = (1 + 3 + 5 … + 99) – (3 + 9 + 15 + 21 + … + 99) – (5 + 25 + 35 + 55 + 65 + 85 + 95) – (17)

$= 2500 - \frac{17}{2}(3+99) - 365 - 17$

= 2500 – 867 – 365 – 17

= 1251

5. Let a be first term and d be common diff. of this A.P.

Given $S_{3n} = 3S_{2n}$

$$\Rightarrow \frac{3n}{2}\left[2a+(3n-1)d\right] = 3\frac{2n}{2}\left[2a+(2n-1)d\right]$$

$\Rightarrow$ 2a + (3n – 1)d = 4a + (4n – 2)d

$\Rightarrow$ 2a + (n – 1)d = 0

$$\text{Now } \frac{S_{4n}}{S_{2n}} = \frac{\frac{4n}{2}\left[2a+(4n-1)d\right]}{\frac{2n}{2}\left[2a+(2n-1)d\right]} = \frac{2\left[\underbrace{2a+(n-1)d}_{=0}+3nd\right]}{\left[\underbrace{2a+(n-1)d}_{=0}+nd\right]}$$

$$= \frac{6nd}{nd} = 6$$

6. $\ell = \left(\underbrace{1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}}_{S}+\ldots\right)^{\log_{0.25}\left(\frac{1}{3}+\frac{1}{3^2}+\ldots\right)}$

$S = 1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\ldots.$

$\frac{S}{3} = \quad \frac{1}{3}+\frac{2}{3^2}+\frac{6}{3^3}+\ldots.$

$- \quad - \quad - \quad -$

$\frac{2S}{3} = 1+\frac{1}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\ldots.$

$\frac{2S}{3} = \frac{4}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\ldots.$

$S = \frac{3}{2}\left(\frac{4/3}{1-1/3}\right) = 3$

Now $\ell = (3)^{\log_{0.25}\left(\frac{1/3}{1-1/3}\right)}$

$\ell = 3^{\log_{((1/4))}\left(\frac{1}{2}\right)} = 3^{1/2} = \sqrt{3}$

$\Rightarrow \ell^2 = 3$

7. $2\log_3(2^x-5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$

Let $2^x = t$

$\log_3(t-5)^2 = \log_3 2\left(t-\frac{7}{2}\right)$

$(t-5)^2 = 2t-7$

$t^2 - 12t + 32 = 0$

$(t-4)(t-8) = 0$

$\Rightarrow 2^x = 4$ or $2^x = 8$

X = 2 (Rejected)

Or x = 3

8. $B - C \equiv \{7,13,19,\ldots 97, \ldots.\}$

Now, $n^2 - n \le 100 \times 100$

$\Rightarrow n(n-1) \le 100 \times 100$

$\Rightarrow A = \{1,2,\ldots., 100\}$.

So, $A \cap (B-C) = \{7,13,19,\ldots,97\}$

Hence, sum = $\frac{16}{2}(7+97) = 832$

9. $S = \frac{1}{x+1}+\frac{2}{x^2+1}+\frac{2^2}{x^4+1}+\ldots.\frac{2^{100}}{x^{2^{100}}+1}$

$S+\frac{1}{1-x} = \frac{1}{1-x}+\frac{1}{x+1}+\ldots.$

$= \frac{2}{1-x^2}+\frac{2}{1+x^2}+\ldots$

$S+\frac{1}{1-x} = \frac{2^{101}}{1-x^{2^{101}}}$

Put x = 2

$S = 1-\frac{2^{101}}{2^{2^{101}}-1}$

Not in option (BONUS)

10. Sum of infinite terms :

$\frac{a}{1-r} = 15$ .... (i)

Series formed by square of terms:

$a^2, a^2r^2, a^2r^4, a^2r^6$ .....

Sum = $\frac{a^2}{1-r^2} = 150$

$\Rightarrow \frac{a}{1-r}.\frac{a}{1+r} = 150 \Rightarrow 15.\frac{a}{1+r} = 150$

$\Rightarrow \frac{a}{1+r} = 10$ ...... (ii)

by (i) and (ii) a = 12 ; $r = \frac{1}{5}$

Now series : ar^2, ar^4, ar^6

Sum = $\frac{ar^2}{1-r^2} = \frac{12.\left(1/25\right)}{1-1/25} = \frac{1}{2}$

11. $c_2 = a_2 + b_2 = a_1 - 3 + 2b_1 = 12$

$a_1 + 2b_1 = 15$ ____(1)

$c_3 = a_3 + b_3 = a_1 - 6 + 4b_1 = 13$

$a_1 + 4b_1 = 19$ ____(2)

from (1) & (2) $b_1 = 2, a_1 = 11$

$$\sum_{k=1}^{10} c_k = \sum_{k=1}^{10}(a_k + b_k) = \sum_{k=1}^{10} a_k + \sum_{k=1}^{10} b_k$$

$$= \frac{10}{2}(2\times 11 + 9\times(-3)) + \frac{2(2^{10}-1)}{2-1}$$

$= 5(22 - 27) + 2(1023)$

$= 2046 - 25 = 2021$

12. $$\frac{2(1+2+3+...+y)}{3(1+2+3+...+y)} = \frac{4}{\log_{10} x}$$

$\Rightarrow \log_{10}x = 6 \Rightarrow x = 10^6$

Now,

$$y = (\log_{10} x) + \left(\log_{10} x^{\frac{1}{3}}\right) + \left(\log_{10} x^{\frac{1}{9}}\right) + ..\infty$$

$$= \left(1 + \frac{1}{3} + \frac{1}{9} + ...\infty\right)\log_{10} x$$

$$= \left(\frac{1}{1-\frac{1}{3}}\right)\log_{10} x = 9$$

So, $(x,y) = (10^6, 9)$

13. $$S = \frac{2^2-1^2}{1^2\times 2^2} + \frac{3^2-2^2}{2^2\times 3^2} + \frac{4^2-3^2}{3^2\times 4^2} + ...$$

$$= \left[\frac{1}{1^2} - \frac{1}{2^2}\right] + \left[\frac{1}{2^2} - \frac{1}{3^2}\right] + \left[\frac{1}{3^2} - \frac{1}{4^2}\right] + ... + \left[\frac{1}{10^2} - \frac{1}{11^2}\right]$$

$$= 1 - \frac{1}{121}$$

$$= \frac{120}{121}$$

14. Let numbers be $\frac{a}{r}$, a, ar $\to$ G.P

$\frac{a}{r}$, 2a, ar $\to$ A.P $\Rightarrow 4a = \frac{a}{r} + ar \Rightarrow r + \frac{1}{r} = 4$

$r = 2 \pm \sqrt{3}$

4^{th} form of G.P $= 3r^2 \Rightarrow ar^2 = 3r^2 \Rightarrow a = 3$

$r = 2 + \sqrt{3}$, $a = 3$, $d = 2a - \frac{a}{r} = 3\sqrt{3}$

$$r^2 - d = (2+\sqrt{3})^2 - 3\sqrt{3}$$

$$= 7 + 4\sqrt{3} - 3\sqrt{3}$$

$$= 7 + \sqrt{3}$$

15. $7 \times 8, 10 \times 10, 13 \times 12, 16 \times 14 ......$

$T_n = (3n + 4)(2n + 6) = 2(3n + 4)(n + 3)$

$= 2(3n^2 + 13n + 12) = 6n^2 + 26n + 24$

$$S_{10} = \sum_{n=1}^{10} T_n = 6\sum_{n=1}^{10} n^2 + 26\sum_{n=1}^{10} n + 24\sum_{n=1}^{10} 1$$

$$= \frac{6(10\times 11\times 21)}{6} + 26\times\frac{10\times 11}{2} + 24\times 10$$

$= 10 \times 11\ (21 + 13) + 240$

$= 3980$

Mean $= \frac{S_{10}}{10} = \frac{3980}{10} = 398$

16. $$\frac{\frac{10}{2}(2a_1 + 9d)}{\frac{p}{2}(2a_1 + (p-1)d)} = \frac{100}{p^2}$$

$(2a_1 + 9d)p = 10(2a_1 + (p - 1)d)$

$9dp = 20a_1 - 2pa_1 + 10d(p - 1)$

$9p = (20 - 2p)\frac{a_1}{d} + 10(p - 1)$

$$\frac{a_1}{d} = \frac{(10-p)}{2(10-p)} = \frac{1}{2}$$

$$\therefore \frac{a_{11}}{a_{10}} = \frac{a_1 + 10d}{a_1 + 9d} = \frac{\frac{1}{2}+10}{\frac{1}{2}+9} = \frac{21}{19}$$

17. A = 4 – digit numbers divisible by 3

A = 1002, 1005, …, 9999.

9999 = 1002 + (n – 1)3

$\Rightarrow$ (n – 1)3 = 8997 $\Rightarrow$ n = 3000

B = 4 – digit numbers divisible by 7

B = 1001, 1008, …, 9996

$\Rightarrow$ 9996 = 1001 + (n – 1)7

$\Rightarrow$ n = 1286

$A \cap B$ = 1008, 1029, …, 9996

9996 = 1008 + (n – 1)21

$\Rightarrow$ n = 429

So, no divisible by either 3 or 7

= 3000 + 1286 – 429 = 3857

total 4-digits numbers = 9000

required numbers = 9000 – 3857 = 5143

18. $$S = \frac{7}{5} + \frac{9}{5^2} + \frac{13}{5^3} + \frac{19}{5^4} + ...$$

$$\frac{1}{5}S = \quad \frac{7}{5^2} + \frac{9}{5^3} + \frac{13}{5^4} + ...$$

On subtracting

$$\frac{4}{5}S = \frac{7}{5} + \frac{2}{5^2} + \frac{4}{5^3} + \frac{6}{5^4} + ...$$

$$S = \frac{7}{4} + \frac{1}{10}\left(1 + \frac{2}{5} + \frac{3}{5^2} + ...\right)$$

$$S = \frac{7}{4} + \frac{1}{10}\left(1 - \frac{1}{5}\right)^{-2}$$

$$= \frac{7}{4} + \frac{1}{10} \times \frac{25}{16} = \frac{61}{32}$$

$\Rightarrow$ 160S = 5 × 61 = 305

19. Let $T_r = r(n-r)$

$T_r = nr - r^2$

$$\Rightarrow S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left(nr - r^2\right)$$

$$S_n = \frac{n.(n)(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$S_n = \frac{n(n-1)(n+1)}{6}$$

Now $$\sum_{r=4}^{\infty}\left(\frac{2S_n}{n!} - \frac{1}{(n-2)!}\right)$$

$$= \sum_{r=4}^{\infty}\left(2.\frac{n(n-1)(n+1)}{6.n(n-1)(n-2)!} - \frac{1}{(n-2)!}\right)$$

$$= \sum_{r=4}^{\infty}\left(\frac{1}{3}\left(\frac{n-2+3}{(n-2)!}\right) - \frac{1}{(n-2)!}\right)$$

$$= \sum_{r=4}^{\infty}\frac{1}{3}\cdot\frac{1}{(n-3)!} = \frac{1}{3}(e-1)$$

Option (1)

20. $\sum_{n=1}^{20}\frac{1}{a_n a_{n+1}}=\sum_{n=1}^{20}\frac{1}{a_n(a_n+d)}$

$=\frac{1}{d}\sum_{n=1}^{20}\left(\frac{1}{a_n}-\frac{1}{a_n+d}\right)$

$\Rightarrow\frac{1}{d}\left(\frac{1}{a_1}-\frac{1}{a_{21}}\right)=\frac{4}{9}$ (Given)

$\Rightarrow\frac{1}{d}\left(\frac{a_{21}-a_1}{a_1a_{21}}\right)=\frac{4}{9}$

$\Rightarrow\frac{1}{d}\left(\frac{a_1+20d-a_1}{a_1a_2}\right)=\frac{4}{9}\Rightarrow a_1a_2=45$ .... (1)

Now sum of first 21 terms = $\frac{21}{2}$ $(2a_1+20d)=189$

$\Rightarrow a_1+10d=9$ ... (2)

For equation (1) & (2) we get

$a_1=3$ & $d=\frac{3}{5}$

OR

$a_1=15$ & $d=-\frac{3}{5}$

So, $a_6.a_{16}=(a_1+5d)(a_1+15d)$

$\Rightarrow a_6a_{16}=72$

Option (2)

21. $\frac{a+2+a}{3}=\frac{10}{3}$

$a=4$

and $\frac{c+b+b}{3}=\frac{7}{3}$

$c+2b=7$

also $2b=a+c$

$2b-a+2b=7$

$b=\frac{11}{4}$

now $4x^2+\frac{11}{4}x+1=0 <^{\alpha}_{\beta}$

$\alpha^2+\beta^2-\alpha\beta=(\alpha+\beta)^2-3\alpha\beta$

$=\left(\frac{-11}{16}\right)^2-3\left(\frac{1}{4}\right)$

$=\frac{121}{256}-\frac{3}{4}=\frac{-71}{256}$

22. Let number are a, ar, ar^2, ar^3

$a\frac{(r^4-1)}{r-1}=\frac{65}{12}$ ...(1)

$\frac{1}{a}\frac{\left(\frac{1}{r^4}-1\right)}{\frac{1}{r}-1}=\frac{65}{18}$

$\frac{1}{ar^3}\left(\frac{1-r^3}{1-r}\right)=\frac{65}{18}$ ....(2)

$\frac{(1)}{(2)}\Rightarrow a^2r^3=\frac{3}{2}$

and $a^3.r^3=1$

$ar=1$

$(ar)^2.r=\frac{3}{2}$

$r=\frac{3}{2}$, $a=\frac{2}{3}$

So, third term = $ar^2=\frac{2}{3}\times\frac{9}{4}$

$\alpha=\frac{3}{2}$

$2\alpha=3$

23. $x = \dfrac{1}{1-\cos^2\theta} \Rightarrow \sin^2\theta = \dfrac{1}{x}$

Also, $\cos^2\theta = \dfrac{1}{y}$ & $1-\sin^2\theta\cos^2\theta = \dfrac{1}{z}$

So, $1-\dfrac{1}{x}\times\dfrac{1}{y}=\dfrac{1}{z} \Rightarrow z(xy-1)=xy$ ....(1)

Also, $\dfrac{1}{x}+\dfrac{1}{y}=1 \quad \Rightarrow x+y=xy$ .....(2)

From (i) and (ii)

$xy + z = xyz = (x + y) z$

24. Let a_n be the side length of A_n.

So, $a_n = \sqrt{2}a_{n+1}$, $a_1 = 12$

$\Rightarrow a_n = 12\times\left(\dfrac{1}{\sqrt{2}}\right)^{n-1}$

Now, $\Rightarrow (a_n)^2 < 1 \qquad \Rightarrow \dfrac{144}{2^{(n-1)}} < 1$

$\Rightarrow 2^{(n-1)} > 144$

$\Rightarrow n - 1 \geq 8$

$\Rightarrow n \geq 9$

25. $T_n = \dfrac{n^2+6n+10}{(2n+1)!} = \dfrac{4n^2+24n+40}{4.(2n+1)!}$

$= \dfrac{(2n+1)^2+20n+39}{4.(2n+1)!}$

$= \dfrac{(2n+1)^2+(2n+1).10+29}{4(2n+1)!}$

$= \dfrac{1}{4}\left[\dfrac{(2n+1)^2}{(2n+1)(2n)!}+\dfrac{(2n+1)10}{(2n+1)(2n)!}+\dfrac{29}{(2n+1)!}\right]$

$= \dfrac{1}{4}\left[\dfrac{2n+1}{(2n)!}+\dfrac{10}{(2n)!}+\dfrac{29}{(2n+1)!}\right]$

$= \dfrac{1}{4}\left[\dfrac{1}{(2n-1)!}+\dfrac{11}{(2n)!}+\dfrac{29}{(2n+1)!}\right]$

$S_1 = \dfrac{1}{1!}+\dfrac{1}{3!}+\dfrac{1}{5!}+... = \dfrac{e-\frac{1}{e}}{2}$

$S_2 = 11\left[\dfrac{1}{2!}+\dfrac{1}{4!}+\dfrac{1}{6!}+...\right] = 11\left[\dfrac{e+\frac{1}{e}-2}{2}\right]$

$S_3 = 29\left[\dfrac{1}{3!}+\dfrac{1}{5!}+\dfrac{1}{7!}+...\right] = 29\left[\dfrac{e-\frac{1}{e}-2}{2}\right]$

Now, $S = \dfrac{1}{4}[S_1+S_2+S_3]$

$= \dfrac{1}{4}\left[\dfrac{e}{2}-\dfrac{1}{2e}+\dfrac{11e}{2}+\dfrac{11}{2e}+\dfrac{29e}{2}-\dfrac{29}{2e}-4\right]$

$= \dfrac{41e}{8}-\dfrac{19}{8e}-10$

26. $4x^2 - 9x + 5 = 0 \Rightarrow x = 1, \dfrac{5}{4}$

Now given $\dfrac{5}{4} = \dfrac{t_p+t_q}{2}, t = t_pt_q$ where

$t_r = -16\left(-\dfrac{1}{2}\right)^{r-1}$

so $\dfrac{5}{4} = -8\left[\left(-\dfrac{1}{2}\right)^{p-1}+\left(-\dfrac{1}{2}\right)^{q-1}\right]$

$1 = 256\left(-\dfrac{1}{2}\right)^{p+q-2} \Rightarrow 2^{p+q-2} = (-1)^{p+q-2}2^8$

hence p + q = 10

27. a, ar, ar^2, ...

$$T_2+T_6=\frac{25}{2}\Rightarrow ar\left(1+r^4\right)=\frac{25}{2}$$

$a^2r^2(1+r^4)^2=\frac{625}{4}$ .... (1)

$T_3 . T_5 = 25 \Rightarrow (ar^2)(ar^4) = 25$

$a^2r^6 = 25$ .... (2)

On dividing (1) by (2)

$$\frac{\left(1+r^4\right)^2}{r^4}=\frac{25}{4}$$

$4r^8 - 17r^4 + 4 = 0$

$(4r^4 - 1)(r^4 - 4) = 0$

$r^4=\frac{1}{4}, 4 \Rightarrow r^4 = 4$

(an increasing geometric series)

$a^2r^6 = 25 \Rightarrow (ar^3)^2 = 25$

$T_4 + T_6 + T_8 = ar^3 + ar^5 + ar^7$

$= ar^3(1 + r^2 + r^4)$

$= 5(1 + 2 + 4) = 35$

28. $S=1+\frac{2}{3}+\frac{7}{3^2}+\frac{12}{3^3}+\frac{17}{3^4}+...$

$$\frac{S}{3}=\frac{1}{3}+\frac{2}{3^2}+\frac{7}{3^3}+\frac{12}{3^4}+...$$

$$\frac{2S}{3}=1+\frac{1}{3}+\frac{5}{3^2}+\frac{5}{3^3}+\frac{5}{3^4}+...+\text{ up to infinite terms}$$

$$\Rightarrow S=\frac{13}{4}$$

29. $a^2=\frac{b}{16}\Rightarrow\frac{1}{b}=\frac{1}{16a^2}$

$$\frac{2}{b}=\frac{1}{a}+6$$

$$\frac{1}{8a^2}=\frac{1}{a}+6$$

$$\frac{1}{a^2}-\frac{8}{a}-48=0$$

$$\frac{1}{a}=12,-4 \Rightarrow a=\frac{1}{12},-\frac{1}{4}$$

$$a=\frac{1}{12},\ a>0$$

$$b=16a^2=\frac{1}{9}$$

$\Rightarrow 72(a+b) = 6 + 8 = 14$

30. $S_n(x) = (2+3+6+11+18+27+.......+\text{n-terms})\log_a x$

Let $S_1 = 2 + 3 + 6 + 11 + 18 + 27 + ... + T_n$

$S_1 = 2 + 3 + 6 + ... + T_n$

$T_n = 2 + 1 + 3 + 5 + ... + \text{n terms}$

$T_n = 2 + (n-1)^2$

$$S_1=\Sigma T_n=2n+\frac{(n-1)n(2n-1)}{6}$$

$$\Rightarrow S_n(x)=\left(2n+\frac{n(n-1)(2n-1)}{6}\right)\log_a x$$

$S_{24}(x) = 1093$ (Given)

$$\log_a x\left(48+\frac{23.24.47}{6}\right)=1093$$

$\log_a x=\frac{1}{4}$ .... (1)

$S_{12}(2x) = 265$

$S_{12}(2x) = 265$

$$\log_a(2x)\left(24+\frac{11.12.23}{6}\right)=265$$

$\log_a 2x=\frac{1}{2}$ .... (2)

(2) – (1)

$$\log_a 2x-\log_a x=\frac{1}{4}$$

$\log_a 2=\frac{1}{4} \Rightarrow a = 16$

31. **GP :** 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192

AP : 11, 16, 21, 26, 31, 36

Common terms : 16, 256, 4096 only

32. $2\log_{10}(4^x-2)=1+\log_{10}\left(4^x+\frac{18}{5}\right)$

$(4^x-2)^2=10\left(4^x+\frac{18}{5}\right)$

$(4^x)^2+4-4(4^x)-32=0$

$(4^x-16)(4^x+2)=0$

$4^x=16$

$x=2$

$\begin{vmatrix}3 & 1 & 4\\1 & 0 & 2\\2 & 1 & 0\end{vmatrix}=3(-2)-1(0-4)+4(1)$

$=-6+4+4=2$

33. $S=(100)(100)+(99)(101)+(98)(102)........$

$(2)(198)+(1)(199)$

$S=\sum_{x=0}^{99}(100-x)(100+x)=\sum 100^2-x^2$

$=100^3-\frac{99\times100\times199}{6}$

$\alpha=3$ $\beta=1650$

slope $=\frac{1650}{3}=550$

34. $T_n=\frac{1}{(2n+1)^2-1}\frac{1}{(2n+2)2n}=\frac{1}{4(n)(n+1)}$

$=\frac{(n+1)-n}{4n(n+1)}=\frac{1}{4}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$S=\frac{1}{4}\left(1-\frac{1}{101}\right)=\frac{1}{4}\left(\frac{100}{101}\right)=\frac{25}{101}$

35. $S_{2n}=\frac{2n}{2}[2a+(2n-1)d],\ S_{4n}=\frac{4n}{2}[2a+(4n-1)d]$

$\Rightarrow S_2-S_1=\frac{4n}{2}[2a+(4n-1)d]-\frac{2n}{2}[2a+(2n-1)d]$

$=4an+(4n-1)2nd-2na-(2n-1)dn$

$=2na+nd[8n-2-2n+1]$

$\Rightarrow 2na+nd[6n-1]=1000$

$2a+(6n-1)d=\frac{1000}{n}$

Now, $S_{6n}=\frac{6n}{2}[2a+(6n-1)d]$

$=3n.\frac{1000}{n}=3000$

36. $\left((x^{1/3}+1)-\left(\frac{\sqrt{x}+1}{\sqrt{x}}\right)\right)^{10}$

$(x^{1/3}-x^{-1/2})^{10}$

$T_{r+1}={}^{10}C_r(x^{1/3})^{10-r}(-x^{-1/2})^r$

$\frac{10-r}{3}-\frac{r}{2}=0\ \Rightarrow 20-2r-3r=0$

$\Rightarrow r=4$

$T_5={}^{10}C_4=\frac{10\times9\times8\times7}{4\times3\times2\times1}=210$