

QUADRATIC EQUATION

1. If α and β are the distinct roots of the equation $x^2 + (3)^{1/4}x + 3^{1/2} = 0$, then the value of $\alpha^{96}(\alpha^{12} - 1) + \beta^{96}(\beta^{12} - 1)$ is equal to :
 (1) 56×3^{25} (2) 56×3^{24}
 (3) 52×3^{24} (4) 28×3^{25}
2. The number of real roots of the equation $e^{6x} - e^{4x} - 2e^{3x} - 12e^{2x} + e^x + 1 = 0$ is :
 (1) 2 (2) 4 (3) 6 (4) 1
3. If α, β are roots of the equation $x^2 + 5(\sqrt{2})x + 10 = 0, \alpha > \beta$ and $P_n = \alpha^n - \beta^n$ for each positive integer n , then the value of $\left(\frac{P_{17}P_{20} + 5\sqrt{2}P_{17}P_{19}}{P_{18}P_{19} + 5\sqrt{2}P_{18}^2}\right)$ is equal to _____.
4. The number of real solutions of the equation, $x^2 - |x| - 12 = 0$ is :
 (1) 2 (2) 3 (3) 1 (4) 4
5. If $a + b + c = 1, ab + bc + ca = 2$ and $abc = 3$, then the value of $a^4 + b^4 + c^4$ is equal to _____.
6. Let α, β be two roots of the equation $x^2 + (20)^{1/4}x + (5)^{1/2} = 0$. Then $\alpha^8 + \beta^8$ is equal to
 (1) 10 (2) 100 (3) 50 (4) 160
7. The number of real roots of the equation $e^{4x} - e^{3x} - 4e^{2x} - e^x + 1 = 0$ is equal to ____.
8. The sum of all integral values of k ($k \neq 0$) for which the equation $\frac{2}{x-1} - \frac{1}{x-2} = \frac{2}{k}$ in x has no real roots, is _____.
9. Let $\lambda \neq 0$ be in \mathbf{R} . If α and β are the roots of the equation $x^2 - x + 2\lambda = 0$, and α and γ are the roots of equation $3x^2 - 10x + 27\lambda = 0$, then $\frac{\beta\gamma}{\lambda}$ is equal to _____.

10. If $x^2 + 9y^2 - 4x + 3 = 0, x, y \in \mathbb{R}$, then x and y respectively lie in the intervals:
 (1) $\left[-\frac{1}{3}, \frac{1}{3}\right]$ and $\left[-\frac{1}{3}, \frac{1}{3}\right]$
 (2) $\left[-\frac{1}{3}, \frac{1}{3}\right]$ and $[1, 3]$
 (3) $[1, 3]$ and $[1, 3]$
 (4) $[1, 3]$ and $\left[-\frac{1}{3}, \frac{1}{3}\right]$
11. The number of distinct real roots of the equation $3x^4 + 4x^3 - 12x^2 + 4 = 0$ is _____.
12. The set of all values of $k > -1$, for which the equation $(3x^2 + 4x + 3)^2 - (k+1)(3x^2 + 4x + 3)(3x^2 + 4x + 2) + k(3x^2 + 4x + 2)^2 = 0$ has real roots, is :
 (1) $\left(1, \frac{5}{2}\right]$ (2) $[2, 3)$
 (3) $\left[-\frac{1}{2}, 1\right)$ (4) $\left(\frac{1}{2}, \frac{3}{2}\right] - \{1\}$
13. $\operatorname{cosec}18^\circ$ is a root of the equation :
 (1) $x^2 + 2x - 4 = 0$ (2) $4x^2 + 2x - 1 = 0$
 (3) $x^2 - 2x + 4 = 0$ (4) $x^2 - 2x - 4 = 0$
14. The numbers of pairs (a, b) of real numbers, such that whenever α is a root of the equation $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root of this equation, is :
 (1) 6 (2) 2 (3) 4 (4) 8
15. The number of the real roots of the equation $(x + 1)^2 + |x - 5| = \frac{27}{4}$ is _____.
16. Let p and q be two positive numbers such that $p + q = 2$ and $p^4 + q^4 = 272$. Then p and q are roots of the equation :
 (1) $x^2 - 2x + 2 = 0$

(2) $x^2 - 2x + 8 = 0$

(3) $x^2 - 2x + 136 = 0$

(4) $x^2 - 2x + 16 = 0$

17. The integer 'k', for which the inequality $x^2 - 2(3k - 1)x + 8k^2 - 7 > 0$ is valid for every x in \mathbb{R} , is :

(1) 3 (2) 2 (3) 0 (4) 4

18. If $\alpha, \beta \in \mathbb{R}$ are such that $1 - 2i$ (here $i^2 = -1$) is a root of $z^2 + \alpha z + \beta = 0$, then $(\alpha - \beta)$ is equal to :

(1) -3 (2) -7 (3) 7 (4) 3

19. Let α and β be the roots of $x^2 - 6x - 2 = 0$. If

$a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{3a_9}$ is:

(1) 2 (2) 1 (3) 4 (4) 3

20. Let α and β be two real numbers such that $\alpha + \beta = 1$ and $\alpha\beta = -1$. Let $p_n = (\alpha)^n + (\beta)^n$, $p_{n-1} = 11$ and $p_{n+1} = 29$ for some integer $n \geq 1$. Then, the value of p_n^2 is _____.

21. The number of solutions of the equation $\log_4(x - 1) = \log_2(x - 3)$ is

22. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be defined as

$f(x) = ax^2 + bx + c$ for all $x \in [-1, 1]$, where $a, b, c \in \mathbb{R}$ such that $f(-1) = 2$, $f'(-1) = 1$ and for $x \in (-1, 1)$ the maximum value of $f''(x)$ is $\frac{1}{2}$.

If $f(x) \leq \alpha$, $x \in [-1, 1]$, then the least value of α is equal to _____.

23. The value of $3 + \frac{1}{4 + \frac{1}{3 + \frac{1}{4 + \frac{1}{3 + \dots \infty}}}}$ is equal to

(1) $1.5 + \sqrt{3}$

(2) $2 + \sqrt{3}$

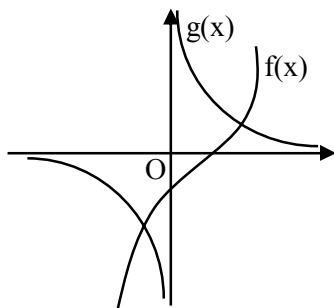
(3) $3 + 2\sqrt{3}$

(4) $4 + \sqrt{3}$

SOLUTION

1. As, $(\alpha^2 + \sqrt{3}) = -(3)^{1/4} \cdot \alpha$
 $\Rightarrow (\alpha^4 + 2\sqrt{3}\alpha^2 + 3) = \sqrt{3}\alpha^2$ (On squaring)
 $\therefore (\alpha^4 + 3) = (-)\sqrt{3}\alpha^2$
 $\Rightarrow \alpha^8 + 6\alpha^4 + 9 = 3\alpha^4$ (Again squaring)
 $\therefore \alpha^8 + 3\alpha^4 + 9 = 0$
 $\Rightarrow \boxed{\alpha^8 = -9 - 3\alpha^4}$
 (Multiply by α^4)
 So, $\alpha^{12} = -9\alpha^4 - 3\alpha^8$
 $\therefore \alpha^{12} = -9\alpha^4 - 3(-9 - 3\alpha^4)$
 $\Rightarrow \alpha^{12} = -9\alpha^4 + 27 + 9\alpha^4$
 Hence, $\boxed{\alpha^{12} = (27)^2}$
 $\Rightarrow (\alpha^{12})^8 = (27)^8$
 $\Rightarrow \alpha^{96} = (3)^{24}$
 Similarly $\beta^{96} = (3)^{24}$
 $\therefore \alpha^{96}(\alpha^{12} - 1) + \beta^{96}(\beta^{12} - 1) = (3)^{24} \times 52$
 \Rightarrow Option (3) is correct.

2. $e^{6x} - e^{4x} - 2e^{3x} - 12e^{2x} + e^x + 1 = 0$
 $\Rightarrow (e^{3x} - 1)^2 - e^x(e^{3x} - 1) = 12e^{2x}$
 $(e^{3x} - 1)^2(e^x - e^{-x} - e^{-2x}) = 12$
 $\Rightarrow \underbrace{e^x - e^{-x} - e^{-2x}}_{\text{increasing (let } f(x))} = \frac{12}{\underbrace{e^{3x} - 1}_{\text{decreasing (let } g(x))}}$



\Rightarrow No. of real roots = 2

3. $x^2 + 5\sqrt{2}x + 10 = 0$
 & $p_n = \alpha^n - \beta^n$ (Given)
 Now $\frac{P_{17}P_{20} + 5\sqrt{2}P_{17}P_{19}}{P_{18}P_{19} + 5\sqrt{2}P_{18}^2} = \frac{P_{17}(P_{20} + 5\sqrt{2}P_{19})}{P_{18}(P_{19} + 5\sqrt{2}P_{18})}$
 $\frac{P_{17}(\alpha^{20} - \beta^{20} + 5\sqrt{2}(\alpha^{19} - \beta^{19}))}{P_{18}(\alpha^{19} - \beta^{19} + 5\sqrt{2}(\alpha^{18} - \beta^{18}))}$
 $\frac{P_{17}(\alpha^{19}(\alpha + 5\sqrt{2}) - \beta^{19}(\beta + 5\sqrt{2}))}{P_{18}(\alpha^{18}(\alpha + 5\sqrt{2}) - \beta^{18}(\beta + 5\sqrt{2}))}$
 Since $\alpha + 5\sqrt{2} = -10/\alpha$ (1)
 & $\beta + 5\sqrt{2} = -10/\beta$ (2)
 Now put these values in above expression
 $= -\frac{10P_{17}P_{18}}{-10P_{18}P_{17}} = 1$
 4. $|x|^2 - |x| - 12 = 0$
 $(|x| + 3)(|x| - 4) = 0$
 $|x| = 4 \Rightarrow x = \pm 2$
 5. $a^2 + b^2 + c^2 = (a + b + c)^2 - 2\Sigma ab = -3$
 $(ab + bc + ca)^2 = \Sigma(ab)^2 + 2abc\Sigma a$
 $\Rightarrow \Sigma(ab)^2 = -2$
 $a^4 + b^4 + c^4 = (a^2 + b^2 + c^2)^2 - 2\Sigma(ab)^2$
 $= 9 - 2(-2) = 13$
 6. $(x^2 + \sqrt{5})^2 = \sqrt{20}x^2$
 $x^4 - 5 = 0 \Rightarrow x^8 = 25$
 $\alpha^8 + \beta^8 = 50$
 7. $t^4 - t^3 - 4t^2 - t + 1 = 0, e^x = t > 0$
 $\Rightarrow t^2 - t - 4 - \frac{1}{t} + \frac{1}{t^2} = 0$
 $\Rightarrow \alpha^2 - \alpha - 6 = 0, \alpha = t + \frac{1}{t} \geq 2$
 $\Rightarrow \alpha = 3, -2$ (reject)
 $\Rightarrow t + \frac{1}{t} = 3$
 \Rightarrow The number of real roots = 2

8. $\frac{2}{x-1} - \frac{1}{x-2} = \frac{2}{k}$
 $x \in \mathbb{R} - \{1, 2\}$
 $\Rightarrow k(2x - 4 - x + 1) = 2(x^2 - 3x + 2)$
 $\Rightarrow k(x - 3) = 2(x^2 - 3x + 2)$
 for $x \neq 3$, $k = 2\left(x - 3 + \frac{2}{x-3} + 3\right)$
 $x - 3 + \frac{2}{x-3} \geq 2\sqrt{2}, \forall x > 3$
 & $x - 3 + \frac{2}{x-3} \leq -2\sqrt{2}, \forall x < -3$
 $\Rightarrow 2\left(x - 3 + \frac{2}{x-3} + 3\right) \in (-\infty, 6 - 4\sqrt{2}] \cup [6 + 4\sqrt{2}, \infty)$
 for no real roots
 $k \in (6 - 4\sqrt{2}, 6 + 4\sqrt{2}) - \{0\}$
 Integral $k \in \{1, 2, \dots, 11\}$
 Sum of $k = 66$
9. $3\alpha^2 - 10\alpha + 27\lambda = 0$ _____(1)
 $\alpha^2 - \alpha + 2\lambda = 0$ _____(2)
 (1) - 3(2) gives
 $-7\alpha + 21\lambda = 0 \Rightarrow \alpha = 3\lambda$
 Put $\alpha = 3\lambda$ in equation (1) we get
 $9\lambda^2 - 3\lambda + 2\lambda - 0$
 $9\lambda^2 = \lambda \Rightarrow \lambda = \frac{1}{9}$ as $\lambda \neq 0$
 Now $\alpha = 3\lambda \Rightarrow \lambda = \frac{1}{3}$
 $\alpha + \beta = 1 \Rightarrow \beta = 2/3$
 $\alpha + \gamma = \frac{10}{3} \Rightarrow \gamma = 3$
 $\frac{\beta\gamma}{\lambda} = \frac{\frac{2}{3} \times 3}{\frac{1}{9}} = 18$

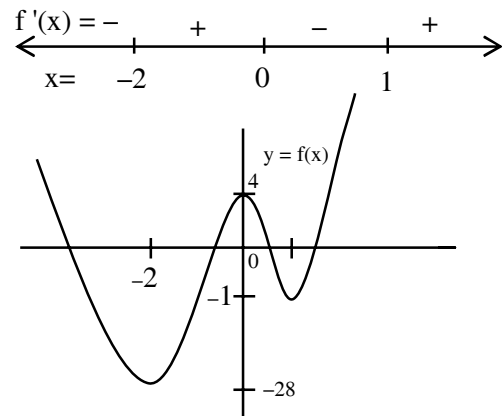
10. $x^2 + 9y^2 - 4x + 3 = 0$
 $(x^2 - 4x) + (9y^2) + 3 = 0$
 $(x^2 - 4x + 4) + (9y^2) + 3 - 4 = 0$
 $(x - 2)^2 + (3y)^2 = 1$
 $\frac{(x-2)^2}{(1)^2} + \frac{y^2}{\left(\frac{1}{3}\right)^2} = 1$ (equation of an ellipse).

As it is equation of an ellipse, x & y can vary inside the ellipse.

$$\text{So, } x - 2 \in [-1, 1] \text{ and } y \in \left[-\frac{1}{3}, \frac{1}{3}\right]$$

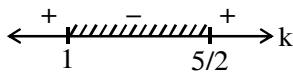
$$x \in [1, 3] \quad y \in \left[-\frac{1}{3}, \frac{1}{3}\right]$$

11. $3x^4 + 4x^3 - 12x^2 + 4 = 0$
 So, Let $f(x) = 3x^4 + 4x^3 - 12x^2 + 4$
 $\therefore f'(x) = 12x(x^2 + x - 2)$
 $= 12x(x + 2)(x - 1)$



12. $(3x^2 + 4x + 3)^2 - (k + 1)(3x^2 + 4x + 3)(3x^2 + 4x + 2) + k(3x^2 + 4x + 2)^2 = 0$
 Let $3x^2 + 4x + 3 = a$
 and $3x^2 + 4x + 2 = b \Rightarrow b = a - 1$
 Given equation becomes
 $\Rightarrow a^2 - (k + 1)ab + kb^2 = 0$
 $\Rightarrow a(a - kb) - b(a - kb) = 0$
 $\Rightarrow (a - kb)(a - b) = 0 \Rightarrow a = kb$ or $a = b$
 (reject)
 $\therefore a = kb$

$\Rightarrow 3x^2 + 4x + 3 = k(3x^2 + 4x + 2)$
 $\Rightarrow 3(k - 1)x^2 + 4(k - 1)x + (2k - 3) = 0$
 for real roots $D \geq 0$
 $\Rightarrow 16(k - 1)^2 - 4(3(k - 1))(2k - 3) \geq 0$
 $\Rightarrow 4(k - 1)\{4(k - 1) - 3(2k - 3)\} \geq 0$
 $\Rightarrow 4(k - 1)\{-2k + 5\} \geq 0$
 $\Rightarrow -4(k - 1)\{2k - 5\} \geq 0$
 $\Rightarrow (k - 1)(2k - 5) \leq 0$



$\therefore k \in \left[1, \frac{5}{2}\right]$

$\therefore k \neq 1$

$\therefore k \in \left(1, \frac{5}{2}\right]$ Ans.

13. $\operatorname{cosec} 18^\circ = \frac{1}{\sin 18^\circ} = \frac{4}{\sqrt{5} - 1} = \sqrt{5} + 1$

Let $\operatorname{cosec} 18^\circ = x = \sqrt{5} + 1$

$\Rightarrow x - 1 = \sqrt{5}$

Squaring both sides, we get

$x^2 - 2x + 1 = 5$

$\Rightarrow x^2 - 2x - 4 = 0$

14. Consider the equation $x^2 + ax + b = 0$

If has two roots (not necessarily real α & β)

Either $\alpha = \beta$ or $\alpha \neq \beta$

Case (1) If $\alpha = \beta$, then it is repeated root. Given that $\alpha^2 - 2$ is also a root

So, $\alpha = \alpha^2 - 2 \Rightarrow (\alpha + 1)(\alpha - 2) = 0$

$\Rightarrow \alpha = -1$ or $\alpha = 2$

When $\alpha = -1$ then $(a, b) = (2, 1)$

$\alpha = 2$ then $(a, b) = (-4, 4)$

Case (2) If $\alpha \neq \beta$ Then

(I) $\alpha = \alpha^2 - 2$ and $\beta = \beta^2 - 2$

Here $(\alpha, \beta) = (2, -1)$ or $(-1, 2)$

Hence $(a, b) = -(\alpha + \beta), \alpha\beta$

$= (-1, -2)$

(II) $\alpha = \beta^2 - 2$ and $\beta = \alpha^2 - 2$

Then $\alpha - \beta = \beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha)$

Since $\alpha \neq \beta$ we get $\alpha + \beta = \beta^2 + \alpha^2 - 4$

$\alpha + \beta = (\alpha + \beta)^2 - 2\alpha\beta - 4$

Thus $-1 = 1 - 2\alpha\beta - 4$ which implies

$\alpha\beta = -1$ Therefore $(a, b) = -(\alpha + \beta), \alpha\beta$

$= (1, -1)$

(III) $\alpha = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$

$\Rightarrow \alpha = -\beta$

Thus $\alpha = 2, \beta = -2$

$\alpha = -1, \beta = 1$

Therefore $(a, b) = (0, -4)$ & $(0, -1)$

(IV) $\beta = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$ is same as (III)

Therefore we get 6 pairs of (a, b)

Which are $(2, 1), (-4, 4), (-1, -2), (1, -1), (0, -4)$

Option (1)

15. Case-I

$$x \leq 5$$

$$(x+1)^2 - (x-5) = \frac{27}{4}$$

$$(x+1)^2 - (x+1) - \frac{3}{4} = 0$$

$$x+1 = \frac{3}{2}, -\frac{1}{2}$$

$$x = \frac{1}{2}, -\frac{3}{2}$$

Case-II

$$x > 5$$

$$(x+1) + (x-5) = \frac{27}{4}$$

$$(x+1)^2 + (x+1) - \frac{51}{4} = 0$$

$$x = \frac{-1 \pm \sqrt{52}}{2} \text{ (rejected as } x > 5)$$

So, the equation have two real root.

16. Consider $(p^2 + q^2)^2 - 2p^2q^2 = 272$
 $((p+q)^2 - 2pq)^2 - 2p^2q^2 = 272$
 $16 - 16pq + 2p^2q^2 = 272$
 $(pq)^2 - 8pq - 128 = 0$

$$(pq)^2 - 8pq - 128 = 0$$

$$pq = \frac{8 \pm 24}{2} = 16, -8$$

$$\therefore pq = 16$$

$$\therefore \text{ Required equation : } x^2 - (2)x + 16 = 0$$

17. $x^2 - 2(3K-1)x + 8K^2 - 7 > 0$

$$\text{Now, } D < 0$$

$$\Rightarrow 4(3K-1)^2 - 4 \times 1 \times (8K^2 - 7) < 0$$

$$\Rightarrow 9K^2 - 6K + 1 - 8K^2 + 7 < 0$$

$$\Rightarrow K^2 - 6K + 8 < 0$$

$$\Rightarrow (K-4)(K-2) < 0$$

$$\Rightarrow \boxed{K \in (2, 4)}$$

18. $\therefore \alpha, \beta \in \mathbb{R} \Rightarrow$ other root is $1 + 2i$

$$\alpha = -(\text{sum of roots}) = -(1 - 2i + 1 + 2i) = -2$$

$$\beta = \text{product of roots} = (1 - 2i)(1 + 2i) = 5$$

$$\therefore \alpha - \beta = -7$$

option (2)

19. $\alpha^2 - 6\alpha - 2 = 0$

$$\alpha^{10} - 6\alpha^9 - 2\alpha^8 = 0$$

Similarly $\beta^{10} - 6\beta^9 - 2\beta^8 = 0$

$$(\alpha^{10} - \beta^{10}) - 6(\alpha^9 - \beta^9) - 2(\alpha^8 - \beta^8) = 0$$

$$\Rightarrow a_{10} - 6a_9 - 2a_8 = 0$$

$$\Rightarrow \frac{a_{10} - 2a_8}{3a_9} = 2$$

20. $x^2 - x - 1 = 0$ roots = α, β

$$\alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha^{n+1} = \alpha^n + \alpha^{n-1}$$

$$\beta^2 - \beta - 1 = 0 \Rightarrow \beta^{n+1} = \beta^n + \beta^{n-1}$$

+

$$P_{n+1} = P_n + P_{n-1}$$

$$29 = P_n + 11$$

$$P_n = 18$$

$$P_n^2 = 324$$

21. $\log_4(x-1) = \log_2(x-3)$

$$\Rightarrow \frac{1}{2} \log_2(x-1) = \log_2(x-3)$$

$$\Rightarrow \log_2(x-1)^{1/2} = \log_2(x-3)$$

$$\Rightarrow (x-1)^{1/2} = x-3$$

$$\Rightarrow x-1 = x^2 + 9 - 6x$$

$$\Rightarrow x^2 - 7x + 10 = 0$$

$$\Rightarrow (x-2)(x-5) = 0$$

$$\Rightarrow x = 2, 5$$

But $x \neq 2$ because it is not satisfying the domain of given equation i.e $\log_2(x-3) \rightarrow$ its domain $x > 3$

finally x is 5

\therefore No. of solutions = 1.

22. $f : [-1, 1] \rightarrow \mathbb{R}$

$$f(x) = ax^2 + bx + c$$

$$f(-1) = a - b + c = 2 \quad \dots(1)$$

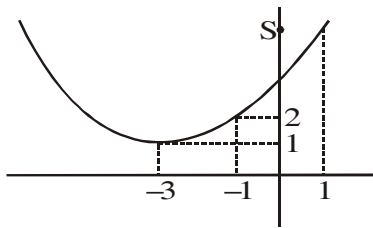
$$f'(-1) = -2a + b = 1 \quad \dots(2)$$

$$f''(x) = 2a$$

$$\Rightarrow \text{Max. value of } f''(x) = 2a = \frac{1}{2}$$

$$\Rightarrow a = \frac{1}{4}; b = \frac{3}{2}; c = \frac{13}{4}$$

$$\therefore f(x) = \frac{x^2}{4} + \frac{3}{2}x + \frac{13}{4}$$



$$\text{For, } x \in [-1, 1] \Rightarrow 2 \leq f(x) \leq 5$$

\therefore Least value of α is 5

24. Let $x = 3 + \frac{1}{4 + \frac{1}{3 + \frac{1}{4 + \frac{1}{3 + \dots \infty}}}}$

$$\text{So, } x = 3 + \frac{1}{4 + \frac{1}{x}} = 3 + \frac{1}{\frac{4x+1}{x}}$$

$$\Rightarrow (x-3) = \frac{x}{4x+1}$$

$$\Rightarrow (4x+1)(x-3) = x$$

$$\Rightarrow 4x^2 - 12x + x - 3 = x$$

$$\Rightarrow 4x^2 - 12x - 3 = 0$$

$$x = \frac{12 \pm \sqrt{(12)^2 + 12 \times 4}}{2 \times 4} = \frac{12 \pm \sqrt{12(16)}}{8}$$

$$= \frac{12 \pm 4 \times 2\sqrt{3}}{8} = \frac{3 \pm 2\sqrt{3}}{2}$$

$$x = \frac{3}{2} \pm \sqrt{3} = 1.5 \pm \sqrt{3}$$

But only positive value is accepted

$$\text{So, } x = 1.5 + \sqrt{3}$$