METHOD OF DIFFERENTIATION

1. Consider the function $f(x)=\frac{P(x)}{\sin(x-2)},\quad x\neq 2$
$=7 \quad , \quad x=2$

Where P(x) is a polynomial such that P″ (x) is always a constant and P(3) = 9. If f(x) is continuous at x = 2, then P(5) is equal to ______.

2. Let f(x) = $\cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right)$, $0 < x < 1$. Then :

(1) $(1-x)^2 f'(x) - 2(f(x))^2 = 0$

(2) $(1+x)^2 f'(x) + 2(f(x))^2 = 0$

(3) $(1-x)^2 f'(x) + 2(f(x))^2 = 0$

(4) $(1+x)^2 f'(x) - 2(f(x))^2 = 0$

3. If y = y(x) is an implicit function of x such that $\log_e(x+y) = 4xy$, then $\frac{d^2y}{dx^2}$ at x = 0 is equal to _______.

4. If $y^{1/4}+y^{-1/4}=2x$, and $(x^2-1)\frac{d^2y}{dx^2}+\alpha x\frac{dy}{dx}+\beta y=0$, then $|\alpha-\beta|$ is equal to __________.

5. If $y(x)=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right), x\in\left(\frac{\pi}{2},\pi\right)$, then $\frac{dy}{dx}$ at $x=\frac{5\pi}{6}$ is :

(1) $-\frac{1}{2}$ (2) -1 (3) $\frac{1}{2}$ (4) 0

6. $\lim_{x\to 0}\frac{\int_0^{x^2}\left(\sin\sqrt{t}\right)dt}{x^3}$ is equal to :

(1) $\frac{2}{3}$ (2) $\frac{3}{2}$ (3) 0 (4) $\frac{1}{15}$

7. Let $f(x)$ be a differentiable function at x = a with $f'(a) = 2$ and $f(a) = 4$. Then $\lim_{x\to a}\frac{xf(a)-af(x)}{x-a}$ equals :

(1) 2a + 4 (2) 4 – 2a (3) 2a – 4 (4) a + 4

8. The maximum slope of the curve $y=\frac{1}{2}x^4-5x^3+18x^2-19x$ occurs at the point

(1) (2,2) (2) (0,0)

(3) (2,9) (4) $\left(3,\frac{21}{2}\right)$

9. Let f be a twice differentiable function defined on R such that f(0) = 1, f '(0) = 2 and f '(x) ≠ 0 for all x ∈ R. If $\begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x)\end{vmatrix}=0$, for all x ∈ R, then the value of f(1) lies in the interval:

(1) (9, 12) (2) (6, 9) (3) (0, 3) (4) (3, 6)

10. Let f : S → S where S = (0, ∞) be a twice differentiable function such that f(x + 1) = xf(x). If g : S → R be defined as $g(x) = \log_e f(x)$, then the value of |g"(5) – g"(1)| is equal to :

(1) $\frac{205}{144}$ (2) $\frac{197}{144}$ (3) $\frac{187}{144}$ (4) 1

11. If $f(x)=\sin\left(\cos^{-1}\left(\frac{1-2^{2x}}{1+2^{2x}}\right)\right)$ and its first derivative with respect to x is $-\frac{b}{a}\log_e 2$ when x = 1, where a and b are integers, then the minimum value of $|a^2 - b^2|$ is _______ .

SOLUTION

1. Official Ans. by NTA (39)

Sol. $$f(x)=\begin{cases}\dfrac{P(x)}{\sin(x-2)}, & x\neq 2\\ 7, & x=2\end{cases}$$

P''(x) = const. ⇒ P(x) is a 2 degree polynomial

$f(x)$ is cont. at x = 2

$f(2^+) = f(2^-)$

$$\lim_{x\to 2^+}\frac{P(x)}{\sin(x-2)}=7$$

$$\lim_{x\to 2^+}\frac{(x-2)(ax+b)}{\sin(x-2)}=7 \Rightarrow \boxed{2a+b=7}$$

P(x) = (x – 2)(ax + b)

$P(3) = (3-2)(3a+b) = 9 \Rightarrow \boxed{3a+b=9}$

$\boxed{a=2,\ b=3}$

P(5) = (5 – 2)(2.5 + 3) = 3.13 = 39

2. Official Ans. by NTA (3)

Sol. $$f(x)=\cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right)$$

$$\cot^{-1}\sqrt{\frac{1-x}{x}}=\sin^{-1}\sqrt{x}$$

or $f(x) = \cos(2\tan^{-1}\sqrt{x})$

$$=\cos\tan^{-1}\left(\frac{2\sqrt{x}}{1-x}\right)$$

$$f(x)=\frac{1-x}{1+x}$$

Now $$f'(x)=\frac{-2}{(1+x)^2}$$

or $$f'(x)(1-x)^2=-2\left(\frac{1-x}{1+x}\right)^2$$

or $(1-x)^2 f'(x) + 2(f(x))^2 = 0.$

3. Official Ans. by NTA (40)

Sol. $\ln(x+y) = 4xy$ (At x = 0, y = 1)

$x + y = e^{4xy}$

$$\Rightarrow 1+\frac{dy}{dx}=e^{4xy}\left(4x\frac{dy}{dx}+4y\right)$$

At x = 0 $\boxed{\dfrac{dy}{dx}=3}$

$$\frac{d^2y}{dx^2}=e^{4xy}\left(4x\frac{dy}{dx}+4y\right)^2+e^{4xy}\left(4x\frac{d^2y}{dx^2}+4y\right)$$

At x = 0, $$\frac{d^2y}{dx^2}=e^0(4)^2+e^0(24)$$

$$\Rightarrow \frac{d^2y}{dx^2}=40$$

4. Official Ans. by NTA (17)

Sol. $$y^{\frac{1}{4}}+\frac{1}{y^{\frac{1}{4}}}=2x$$

$$\Rightarrow \left(y^{\frac{1}{4}}\right)^2-2xy^{\left(\frac{1}{4}\right)}+1=0$$

$$\Rightarrow y^{\frac{1}{4}}=x+\sqrt{x^2-1} \text{ or } x-\sqrt{x^2-1}$$

So, $$\frac{1}{4}\frac{1}{y^{\frac{3}{4}}}\frac{dy}{dx}=1+\frac{x}{\sqrt{x^2-1}}$$

$$\Rightarrow \frac{1}{4}\frac{1}{y^{3/4}}\frac{dy}{dx}=\frac{y^{\frac{1}{4}}}{\sqrt{x^2-1}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{4y}{\sqrt{x^2-1}} \quad \ldots(1)$$

Hence, $$\frac{d^2y}{dx^2}=4\frac{\left(\sqrt{x^2-1}\right)y'-\dfrac{yx}{\sqrt{x^2-1}}}{x^2-1}$$

$$\Rightarrow (x^2-1)y''=4\frac{(x^2-1)y'-xy}{\sqrt{x^2-1}}$$

$$\Rightarrow (x^2-1)y''=4\left(\sqrt{x^2-1}y'-\frac{xy}{\sqrt{x^2-1}}\right)$$

$$\Rightarrow (x^2-1)y''=4\left(4y-\frac{xy'}{4}\right) \text{ (from I)}$$

$$\Rightarrow (x^2-1)y''+xy'-16y=0$$

So, |α– β| = 17

5. Official Ans. by NTA (1)

Sol. $y(x) = \cot^{-1}\left[\dfrac{\cos\frac{x}{2}+\sin\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}\right]$

$y(x) = \cot^{-1}\left(\tan\frac{x}{2}\right) = \frac{\pi}{2} - \frac{x}{2}$

$y'(x) = \frac{-1}{2}$

6. Official Ans. by NTA (1)

Sol. $\lim\limits_{x\to 0^+}\dfrac{\int_0^{x^2}\sin\sqrt{t}\,dt}{x^3} = \lim\limits_{x\to 0^+}\dfrac{(\sin x)2x}{3x^2}$

$= \lim\limits_{x\to 0^+}\left(\frac{\sin x}{x}\right)\times\frac{2}{3} = \frac{2}{3}$

7. Official Ans. by NTA (2)

Sol. $f'(a) = 2,\ f(a) = 4$

$\lim\limits_{x\to a}\dfrac{xf(a)-af(x)}{x-a}$

$\Rightarrow \lim\limits_{x\to a}\dfrac{f(a)-af'(x)}{1}$ (Lopitals rule)

$= f(a) - af'(a)$

$= 4 - 2a$

8. Official Ans. by NTA (1)

Sol. $\frac{dy}{dx} = 2x^3 - 15x^2 + 36x - 19$

Since, slope is maximum so,

$\frac{d^2y}{dx^2} = 6x^2 - 30x + 36 = 0$

$\Rightarrow x^2 - 5x + 6 = 0$ | $\frac{d^3y}{dx^3} = 12x - 30$

$x = 2, 3$ | at $x = 2$, $\frac{d^3y}{dx^3} < 0$

So, maxima

at $x = 2$

$y = \frac{1}{2}\times 16 - 5\times 8 + 18\times 4 - 19\times 2$

$= 8 - 40 + 72 - 38 = 80 - 78 = 2$

point (2, 2)

9. Official Ans. by NTA (2)

Sol. $f(x)f''(x) - (f'(x))^2 = 0$

$\frac{f''(x)}{f'(x)} = \frac{f'(x)}{f(x)}$

$\ln(f'(x)) = \ln f(x) + \ln c$

$f'(x) = cf(x)$

$\frac{f'(x)}{f(x)} = c$

$\ln f(x) = cx + k_1$

$f(x) = ke^{cx}$

$f(0) = 1 = k$

$f'(0) = c = 2$

$f(x) = e^{2x}$

$f(1) = e^2 \in (6, 9)$

10. Official Ans by NTA (1)

Sol. $\ln f(x+1) = \ln(xf(x))$

$\ln f(x+1) = \ln x + \ln f(x)$

$\Rightarrow g(x+1) = \ln x + g(x)$

$\Rightarrow g(x+1) - g(x) = \ln x$

$\Rightarrow g''(x+1) - g''(x) = -\frac{1}{x^2}$

Put x = 1, 2, 3, 4

$g''(2) - g''(1) = -\frac{1}{1^2}$ ...(1)

$g''(3) - g''(2) = -\frac{1}{2^2}$ ...(2)

$g''(4) - g''(3) = -\frac{1}{3^2}$ ...(3)

$g''(5) - g''(4) = -\frac{1}{4^2}$ ...(4)

Add all the equation we get

$g''(5) - g''(1) = -\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2}$

$|g''(5) - g''(1)| = \frac{205}{144}$

11. Official Ans. by NTA (481)

Sol. $f(x)=\sin\left(\cos^{-1}\left(\frac{1-2^{2x}}{1+2^{2x}}\right)\right)$ at x= 1 ; $2^{2x}=4$

for $\sin\left(\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right)$;

Let $\tan^{-1}x=\theta$; $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\therefore\ \sin\left(\cos^{-1}\cos 2\theta\right)=\sin 2\theta$

$$\left\{\begin{array}{l}\text{If}\quad x>1\Rightarrow\frac{\pi}{2}>\theta>\frac{\pi}{4}\\ \therefore\qquad \pi>2\theta>\frac{\pi}{2}\end{array}\right\}$$

$=2\sin\theta\cos\theta=\frac{2\tan\theta}{1+\tan^2\theta}$

$=\frac{2x}{1+x^2}$

Hence, $f(x)=\frac{2\cdot 2^x}{1+2^{2x}}$

$\therefore\ f'(x)=\frac{(1+2^{2x})(2.2^x\ln 2)-2^{2x}\cdot 2\cdot\ln 2\cdot 2\cdot 2^x}{(1+2^{2x})}$

$\therefore\ f^1(1)=\frac{20\ln 2-32\ln 2}{25}=-\frac{12}{25}\ln 2$

So, a = 25, b = 12 $\Rightarrow |a^2-b^2|=25^2-12^2$

$=625-144$

$=481$