INDEFINITE INTEGRATION

1. If $\int \frac{dx}{(x^2+x+1)^2} = a\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + b\left(\frac{2x+1}{x^2+x+1}\right) + C$, x > 0 where C is the constant of integration, then the value of $9(\sqrt{3}a+b)$ is equal to _______.

2. If $\int \frac{2e^x+3e^{-x}}{4e^x+7e^{-x}}dx = \frac{1}{14}(ux + v\log_e(4e^x+7e^{-x})) + C$, where C is a constant of integration, then u + v is equal to __________.

3. The integral $\int \frac{1}{\sqrt[4]{(x-1)^3(x+2)^5}}dx$ is equal to :

(where C is a constant of integration)

(1) $\frac{3}{4}\left(\frac{x+2}{x-1}\right)^{\frac{1}{4}} + C$ (2) $\frac{3}{4}\left(\frac{x+2}{x-1}\right)^{\frac{5}{4}} + C$

(3) $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{\frac{1}{4}} + C$ (4) $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{\frac{5}{4}} + C$

4. If $\int \frac{\sin x}{\sin^3 x + \cos^3 x}dx =$

$\alpha\log_e|1+\tan x| + \beta\log_e|1-\tan x+\tan^2 x| + \gamma\tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right) + C,$

when C is constant of integration, then the value of $18(\alpha+\beta+\gamma^2)$ is _______.

5. If $\int \frac{\cos x - \sin x}{\sqrt{8-\sin 2x}}dx = a\sin^{-1}\left(\frac{\sin x+\cos x}{b}\right) + c$, where c is a constant of integration, then the ordered pair (a, b) is equal to :

(1) (–1, 3) (2) (3, 1)
(3) (1, 3) (4) (1, –3)

6. The value of the integral

$$\int \frac{\sin\theta.\sin 2\theta(\sin^6\theta+\sin^4\theta+\sin^2\theta)\sqrt{2\sin^4\theta+3\sin^2\theta+6}}{1-\cos 2\theta}d\theta$$

is :

(where c is a constant of integration)

(1) $\frac{1}{18}\left[11-18\sin^2\theta+9\sin^4\theta-2\sin^6\theta\right]^{\frac{3}{2}} + c$

(2) $\frac{1}{18}\left[9-2\cos^6\theta-3\cos^4\theta-6\cos^2\theta\right]^{\frac{3}{2}} + c$

(3) $\frac{1}{18}\left[9-2\sin^6\theta-3\sin^4\theta-6\sin^2\theta\right]^{\frac{3}{2}} + c$

(4) $\frac{1}{18}\left[11-18\cos^2\theta+9\cos^4\theta-2\cos^6\theta\right]^{\frac{3}{2}} + c$

7. The integral $\int \frac{e^{3\log_e 2x}+5e^{2\log_e 2x}}{e^{4\log_e x}+5e^{3\log_e x}-7e^{2\log_e x}}dx$, x > 0, is equal to : (where c is a constant of integration)

(1) $\log_e|x^2+5x-7|+c$

(2) $4\log_e|x^2+5x-7|+c$

(3) $\frac{1}{4}\log_e|x^2+5x-7|+c$

(4) $\log_e\sqrt{x^2+5x-7}+c$

8. For real numbers α, β, γ and δ, if

$$\int \frac{(x^2-1)+\tan^{-1}\left(\frac{x^2+1}{x}\right)}{(x^4+3x^2+1)\tan^{-1}\left(\frac{x^2+1}{x}\right)}dx$$

$$= \alpha\log_e\left(\tan^{-1}\left(\frac{x^2+1}{x}\right)\right)$$

$$+\beta\tan^{-1}\left(\frac{\gamma(x^2-1)}{x}\right) + \delta\tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

where C is an arbitrary constant, then the value of $10(\alpha+\beta\gamma+\delta)$ is equal to __________.

9. The integral $\int \frac{(2x-1)\cos\sqrt{(2x-1)^2+5}}{\sqrt{4x^2-4x+6}}dx$ is equal to (where c is a constant of integration)

(1) $\frac{1}{2}\sin\sqrt{(2x-1)^2+5}+c$

(2) $\frac{1}{2}\cos\sqrt{(2x+1)^2+5}+c$

(3) $\frac{1}{2}\cos\sqrt{(2x-1)^2+5}+c$

(4) $\frac{1}{2}\sin\sqrt{(2x+1)^2+5}+c$

10. If $f(x) = \int \frac{5x^8+7x^6}{(x^2+1+2x^7)^2}dx, (x \geq 0), f(0) = 0$ and $f(1) = \frac{1}{K}$, then the value of K is

SOLUTION

1. Official Ans. by NTA (15)

Sol. $I=\int \frac{dx}{\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right]^2}$

$\int \frac{dt}{\left(t^2+\frac{3}{4}\right)^2}\quad\left(\text{Put } x+\frac{1}{2}=t\right)$

$=\frac{\sqrt{3}}{2}\int \frac{\sec^2\theta\, d\theta}{\frac{9}{16}\sec^4\theta}\left(\text{Put } t=\frac{\sqrt{3}}{2}\tan\theta\right)$

$=\frac{4\sqrt{3}}{9}\int(1+\cos 2\theta)d\theta$

$=\frac{4\sqrt{3}}{9}\left[\theta+\frac{\sin 2\theta}{2}\right]+c$

$=\frac{4\sqrt{3}}{9}\left[\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{\sqrt{3}(2x+1)}{3+(2x+1)^2}\right]+c$

$=\frac{4\sqrt{3}}{9}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{1}{3}\left(\frac{2x+1}{x^2+x+1}\right)+c$

Hence, $9\left(\sqrt{3}a+b\right)=15$

2. Official Ans. by NTA (7)

Sol. $\int \frac{2e^x}{4e^x+7e^{-x}}dx \;+3\int \frac{e^{-x}}{4e^x+7e^{-x}}dx$

$=\int \frac{2e^{2x}}{4e^{2x}+7}dx+3\int \frac{e^{-2x}}{4+7e^{-2x}}dx$

Let $4e^{2x}+7=T$ Let $4+7e^{-2x}=t$

$8e^{2x}dx=dT$ $-14e^{-2x}dx=dt$

$2e^{2x}dx=\frac{dT}{4}$ $e^{-2x}dx=-\frac{dt}{14}$

$\int \frac{dT}{4T}-\frac{3}{14}\int \frac{dt}{t}$

$=\frac{1}{4}\log T-\frac{3}{14}\log t+C$

$=\frac{1}{4}\log(4e^{2x}+7)-\frac{3}{14}\log\left(4+7e^{-2x}\right)+C$

$=\frac{1}{14}\left[\frac{1}{2}\log\left(4e^x+7e^{-x}\right)+\frac{13}{2}x\right]+C$

$u=\frac{13}{2}, v=\frac{1}{2}\Rightarrow u+v=7$

Aliter :

$2e^x+3e^{-x}=A(4e^x+7e^{-x})+B(4e^x-7e^{-x})+\lambda$

$2=4A+4B$; $3=7A-7B$; $\lambda=0$

$A+B=\frac{1}{2}$

$A-B=\frac{3}{7}$

$A=\frac{1}{2}\left(\frac{1}{2}+\frac{3}{7}\right)=\frac{7+6}{28}=\frac{13}{28}$

$B=A-\frac{3}{7}=\frac{13}{28}-\frac{3}{7}=\frac{13-12}{28}=\frac{1}{28}$

$\int\frac{13}{28}dx+\frac{1}{28}\int\frac{4e^x-7e^{-x}}{4e^x+7e^{-x}}dx$

$\frac{13}{28}x+\frac{1}{28}\ell n|4e^x+7e^{-x}|+C$

$u=\frac{13}{2}; v=\frac{1}{2}$

$\Rightarrow u+v=7$

3. Official Ans. by NTA (3)

Sol. $\int \frac{dx}{(x-1)^{3/4}(x+2)^{5/4}}$

$=\int \frac{dx}{\left(\frac{x+2}{x-1}\right)^{5/4}.(x-1)^2}$

put $\frac{x+2}{x-1}=t$

$=-\frac{1}{3}\int\frac{dt}{t^{5/4}}$

$=\frac{4}{3}.\frac{1}{t^{1/4}}+C$

$=\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4}+C$

4. Official Ans. by NTA (3)

Sol. $=\int \frac{\frac{\sin x}{\cos^3 x}}{1+\tan^3 x}dx=\int \frac{\tan x.\sec^2 x}{(\tan x+1)(1+\tan^2 x-\tan x)}dx$

Let tanx = t ⇒ $\sec^2 x.dx = dt$

$=\int \frac{t}{(t+1)(t^2-t+1)}dt$

$=\int\left(\frac{A}{t+1}+\frac{B(2t-1)}{t^2-t+1}+\frac{C}{t^2-t+1}\right)dx$

$\Rightarrow A(t^2 - t + 1) + B(2t - 1)(t^2 - t + 1) + C(t + 1) = t$

$\Rightarrow t^2(A + 2B) + t(-A + B + C) + A - B + C = 1$

$\therefore A + 2B = 0$ ...(1)

$-A + B + C = 1$ ...(2)

$A - B + C = 0$ ...(3)

$\Rightarrow C=\frac{1}{2} \Rightarrow A-B=-\frac{1}{2}$ ...(4)

$A + 2B = 0$

$A-B=-\frac{1}{2}$

$\Rightarrow 3B=\frac{1}{2} \Rightarrow B=\frac{1}{6}$

$A=-\frac{1}{3}$

$I=-\frac{1}{3}\int\frac{dt}{1+t}+\frac{1}{6}\int\frac{2t-1}{t^2-t+1}dt+\frac{1}{2}\int\frac{dt}{t^2-t+1}$

$=-\frac{1}{3}\ell n|(1+\tan x)|+\frac{1}{6}\ell n|\tan^2 x-\tan x+1|+\frac{1}{2}.\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\left(\tan x-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}\right)$

$=-\frac{1}{3}\ell n|(1+\tan x)|+\frac{1}{6}\ell n|\tan^2 x-\tan x+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right)+C$

$\alpha=-\frac{1}{3},\ \beta=\frac{1}{6},\ \gamma=\frac{1}{\sqrt{3}}$

$18(\alpha+\beta+\gamma^2)=18\left(-\frac{1}{3}+\frac{1}{6}+\frac{1}{3}\right)=3$

5. Official Ans. by NTA (3)

Sol. $\int\frac{\cos x-\sin x}{\sqrt{8-\sin 2x}}dx$

$=\int\frac{\cos x-\sin x}{\sqrt{9-(\sin x+\cos x)^2}}dx$

Let sinx + cosx = t

$\int\frac{dt}{\sqrt{9-t^2}}=\sin^{-1}\frac{t}{3}+c$

$=\sin^{-1}\left(\frac{\sin x+\cos x}{3}\right)+c$

So a = 1, b = 3.

6. Official Ans. by NTA (4)

Sol. $I=\int\frac{\sin\theta.\sin 2\theta(\sin^6\theta+\sin^4\theta+\sin^2\theta)\sqrt{2\sin^4\theta+3\sin^2\theta+6}}{1-\cos 2\theta}d\theta$

$\Rightarrow I=\int\frac{\sin\theta.2\sin\theta\cos\theta.\sin^2\theta(\sin^4\theta+\sin^2\theta+1)(2\sin^4\theta+3\sin^2\theta+6)^{1/2}}{2\sin^2\theta}d\theta$

$=\int\sin^2\theta.\cos\theta(\sin^4\theta+\sin^2\theta+1)(2\sin^4\theta+3\sin^2\theta+6)^{1/2}d\theta$

Let sinθ = t ⇒ cosθ dθ = dt

$\therefore I=\int t^2(t^4+t^2+1)(2t^4+3t^2+6)^{1/2}dt$

$=\int(t^5+t^3+t)t(2t^4+3t^2+6)^{1/2}dt$

$=\int(t^5+t^3+t)(t^2)^{1/2}(2t^4+3t^2+6)^{1/2}dt$

$=\int(t^5+t^3+t)(2t^6+3t^4+6t^2)^{1/2}dt$

Let $2t^6 + 3t^4 + 6t^2 = u^2$

$\Rightarrow 12(t^5 + t^3 + t)\,dt = 2u\,du$

$\therefore I=\int(u^2)^{1/2}\cdot\frac{2u\,du}{12}$

$=\int\frac{u^2}{6}du=\frac{u^3}{18}+C$

$=\frac{(2t^6+3t^4+6t^2)^{3/2}}{18}+C$

when t = sinθ

and $t^2 = 1 - \cos^2\theta$ will give option (4)

7. Official Ans. by NTA (2)

Sol. $\int \frac{e^{3\log_e 2x}+5e^{2\log_e 2x}}{e^{4\log_e x}+5e^{3\log_e x}-7e^{2\log_e x}}dx, x>0$

$= \int \frac{(2x)^3+5(2x)^2}{x^4+5x^3-7x^2}dx = \int \frac{4x^2(2x+5)}{x^2(x^2+5x-7)}dx$

$= 4\int \frac{d(x^2+5x-7)}{(x^2+5x-7)} = 4\log_e|x^2+5x-7|+c$

option (2)

8. Official Ans by NTA (6)

Sol. $\int \frac{(x^2-1)dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)} + \int \frac{dx}{x^4+3x^2+1}$

$\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(\left(x+\frac{1}{x}\right)^2+1\right)\tan^{-1}\left(x+\frac{1}{x}\right)} + \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)dx}{x^4+3x^2+1}$

Put $\tan^{-1}\left(x+\frac{1}{x}\right)=t$

$\int \frac{dt}{t} + \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+5} - \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2+1}$

Put $x-\frac{1}{x}=y$, $x+\frac{1}{x}=z$

$\log_e t + \frac{1}{2}\int \frac{dy}{y^2+5} - \frac{1}{2}\int \frac{dz}{z^2+1}$

$= \log_e \tan^{-1}\left(x+\frac{1}{x}\right) + \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right)$

$-\frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{x}\right)+C$

$\alpha = 1, \beta = \frac{1}{2\sqrt{5}}, \gamma = \frac{1}{\sqrt{5}}, \delta = \frac{-1}{2}$

or

$\alpha = 1, \beta = \frac{-1}{2\sqrt{5}}, \gamma = \frac{-1}{\sqrt{5}}, \delta = \frac{-1}{2}$

$10(\alpha + \beta\gamma + \delta) = 10\left(1+\frac{1}{10}-\frac{1}{2}\right) = 6$

9. Official Ans. by NTA (1)

Sol. $\int \frac{(2x-1)\cos\sqrt{(2x-1)^2+5}}{\sqrt{(2x-1)^2+5}}dx$

$(2x-1)^2+5=t^2$

$2(2x-1)\,2dx = 2t\,dt$

$2\sqrt{t^2-5}dx = t\,dt$

So $\int \frac{\sqrt{t^2-5}\cos t}{2\sqrt{t^2-5}}dt = \frac{1}{2}\sin t + c$

$= \frac{1}{2}\sin\sqrt{(2x-1)^2+5}+c$

10. Official Ans. by NTA (4)

Sol. $f(x) = \int \frac{(5x^8+7x^6)dx}{x^{14}(x^{-5}+x^{-7}+2)^2}$

Let $x^{-5}+x^{-7}+2=t$

$(-5x^{-6}-7x^{-8})dx = dt$

$\Rightarrow f(x) = \int -\frac{dt}{t^2} = \frac{1}{t}+c$

$f(x) = \frac{x^7}{x^2+1+2x^7}$

$f(1) = \frac{1}{4}$