DIFFERENTIABILITY

1. Let a function $g : [0, 4] \to \mathbf{R}$ be defined as

$$g(x)=\begin{cases}\max\limits_{0\le t\le x}\{t^3-6t^2+9t-3\}, & 0\le x\le 3\\ 4-x, & 3<x\le 4\end{cases},$$

then the number of points in the interval (0, 4) where g(x) is NOT differentiable, is _______.

2. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as

$$f(x)=\begin{cases}\dfrac{x^3}{(1-\cos 2x)^2}\log_e\left(\dfrac{1+2xe^{-2x}}{(1-xe^{-x})^2}\right), & x\ne 0\\ \alpha, & x=0\end{cases}.$$

If f is continuous at x = 0, then α is equal to :

(1) 1 (2) 3 (3) 0 (4) 2

3. Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined as

$$f(x)=\begin{cases}3\left(1-\dfrac{|x|}{2}\right) & \text{if } |x|\le 2\\ 0 & \text{if } |x|>2\end{cases}$$

Let $g : \mathbf{R} \to \mathbf{R}$ be given by $g(x) = f(x + 2) - f(x - 2)$. If n and m denote the number of points in **R** where g is not continuous and not differentiable, respectively, then n + m is equal to _______.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that f(2) = 4 and $f'(2)=1$. Then, the value of

$$\lim_{x\to 2}\frac{x^2f(2)-4f(x)}{x-2}$$ is equal to :

(1) 4 (2) 8 (3) 16 (4) 12

5. Let $f : [0, 3] \to \mathbf{R}$ be defined by

f(x) = min {x – [x], 1 + [x] – x}

where [x] is the greatest integer less than or equal to x. Let P denote the set containing all $x \in [0, 3]$ where f is discontinuous, and Q denote the set containing all $x \in (0, 3)$ where f is not differentiable. Then the sum of number of elements in P and Q is equal to _________.

माना $f : [0, 3] \to \mathbf{R}$

6. Let $f : [0, \infty) \to [0, 3]$ be a function defined by

$$f(x)=\begin{cases}\max\{\sin t : 0\le t\le x\}, & 0\le x\le \pi\\ 2+\cos x, & x>\pi\end{cases}$$

Then which of the following is true ?

(1) f is continuous everywhere but not differentiable exactly at one point in $(0, \infty)$

(2) f is differentiable everywhere in $(0, \infty)$

(3) f is not continuous exactly at two points in $(0, \infty)$

(4) f is continuous everywhere but not differentiable exactly at two points in $(0, \infty)$

7. Let [t] denote the greatest integer less than or equal to t. Let f(x) = x – [x], g(x) = 1 – x + [x], and h(x) = min{f(x), g(x)}, $x \in [-2, 2]$. Then h is :

(1) continuous in [–2, 2] but not differentiable at more than four points in (–2, 2)

(2) not continuous at exactly three points in [–2, 2]

(3) continuous in [–2, 2] but not differentiable at exactly three points in (–2, 2)

(4) not continuous at exactly four points in [–2, 2]

8. The function $f(x)=|x^2-2x-3|\cdot e^{|9x^2-12x+4|}$ is not differentiable at exactly :

(1) four points (2) three points

(3) two points (4) one point

9. The number of points, at which the function $f(x) = |2x + 1| - 3|x + 2| + |x^2 + x - 2|$, $x \in R$ is not differentiable, is _______.

10. A function f is defined on [–3, 3] as

$$f(x)=\begin{cases}\min\{|x|,2-x^2\}, & -2\le x\le 2\\ [|x|] & ,2<|x|\le 3\end{cases}$$

where [x] denotes the greatest integer ≤ x. The number of points, where f is not differentiable in (–3, 3) is ______.

11. Let the functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be defined as :

$$f(x)=\begin{cases}x+2, & x<0\\ x^2, & x\ge 0\end{cases} \text{ and } g(x)=\begin{cases}x^3, & x<1\\ 3x-2, & x\ge 1\end{cases}$$

Then, the number of points in $\mathbb{R}$ where $(f\circ g)(x)$ is NOT differentiable is equal to :

(1) 3 (2) 1 (3) 0 (4) 2

12. If $f(x)=\begin{cases}\dfrac{1}{|x|} & ; |x|\ge 1\\ ax^2+b & ; |x|<1\end{cases}$ is differentiable at every point of the domain, then the values of a and b are respectively :

(1) $\frac{1}{2},\frac{1}{2}$ (2) $\frac{1}{2},-\frac{3}{2}$

(3) $\frac{5}{2},-\frac{3}{2}$ (4) $-\frac{1}{2},\frac{3}{2}$

13. Let $f : R \to R$ satisfy the equation $f(x + y) = f(x).f(y)$ for all $x, y \in R$ and $f(x) \ne 0$ for any $x \in R$. If the function f is differentiable at $x = 0$ and $f'(0) = 3$, then $\lim_{h\to 0}\frac{1}{h}(f(h)-1)$ is equal to ______.

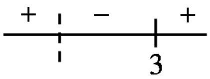
SOLUTION

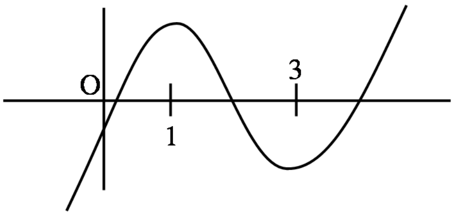
1. Official Ans. by NTA (1)

Sol. $f(x) = x^3 - 6x^2 + 9x - 3$

$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$

$f(1) = 1 \; f(3) = -3$





$$g(x) = \begin{cases} f(x) & 0 \le x \le 1 \\ 0 & 1 \le x \le 3 \\ -1 & 3 < x \le 4 \end{cases}$$

g(x) is continuous

$$g'(x) = \begin{cases} 3(x-1)(x-3) & 0 \le x \le 1 \\ 0 & 1 \le x \le 3 \\ -1 & 3 < x \le 4 \end{cases}$$

g(x) is non-differentiable at x = 3

2. Official Ans. by NTA (1)

Sol. For continuity

$$\lim_{x \to 0} \frac{x^3}{4\sin^4 x}\left(\ell n\left(1 + 2xe^{-2x}\right) - 2\ell n\left(1 - xe^{-x}\right)\right)$$

$= \alpha$

$$\lim_{x \to 0} \frac{1}{4x}\left[2xe^{-2x} + 2xe^{-x}\right] = \alpha$$

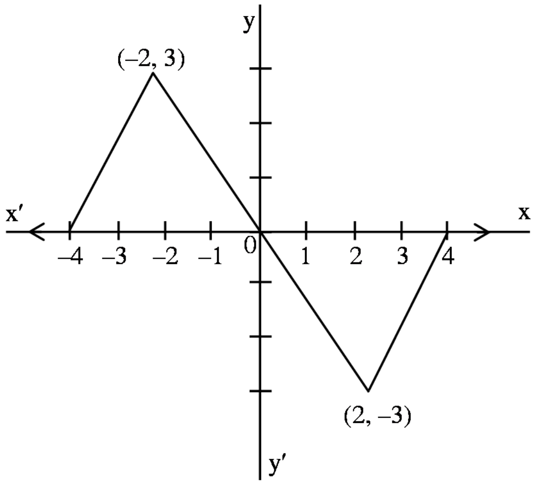
$$= \frac{1}{4}(4) = \alpha = 1$$

3. Official Ans. by NTA (4)

Sol. $$f(x-2) \begin{cases} \frac{3x}{2} & -4 \le x \le -2 \\ -\frac{3x}{2} & -2 < x \le 0 \\ 0 & x \in (-\infty, -4) \cup (0, +\infty) \end{cases}$$

$$f(x-2) \begin{cases} \frac{3x}{2} & 0 \le x \le 2 \\ -\frac{3x}{2} + 6 & 2 \le x \le 4 \\ 0 & x \in (-\infty, 0) \cup (4, +\infty) \end{cases}$$

$$g(x) = f(x+2) - f(x-2) \begin{cases} \frac{3x}{2} + 6 & -4 \le x \le -2 \\ -\frac{3x}{2} & -2 < x < 2 \\ \frac{3x}{2} - 6 & 2 \le x \le 4 \\ 0 & x \in (-\infty, -4) \cup (4, \infty) \end{cases}$$



n = 0

m = 4 ⇒ (n + m = 4)

4. Official Ans. by NTA (4)

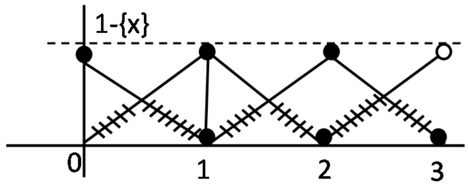
Sol. Apply L'Hopital Rule

$$\lim_{x \to 2}\left(\frac{2xf(2) - 4f'(x)}{1}\right)$$

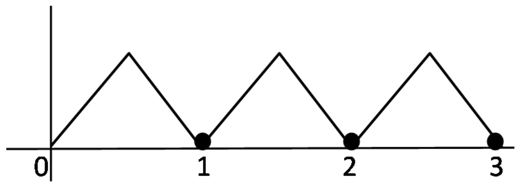
$$= \frac{4(4) - 4}{1} = 12$$

5. Official Ans. by NTA (5)

Sol.



$1 - \{x\} = 1 - x; 0 \le x < 1$

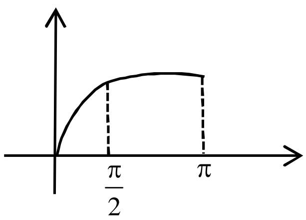


Non differentiable at

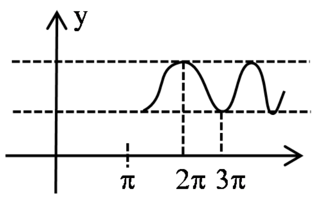
$x = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$

6. Official Ans. by NTA (2)

Sol. Graph of $\max\{\sin t : 0 \le t \le x\}$ in $x \in [0, \pi]$

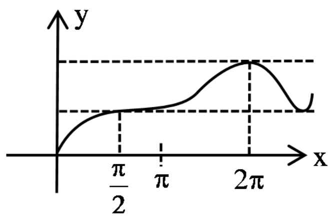


& graph of cos for $x \in [\pi, \infty)$



So graph of

$$f(x) = \begin{cases} \max\{\sin t : 0 \le t \le x, & 0 \le x \le \pi \\ 2 + \cos x & x > h \end{cases}$$

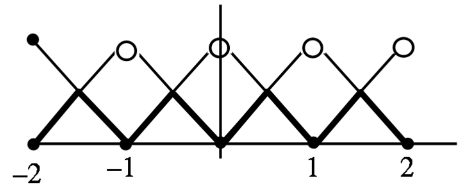


f(x) is differentiable everywhere in $(0, \infty)$

7. Official Ans. by NTA (1)

Sol. min{x – [x], 1 – x + [x]}

h(x) = min{x – [x], 1 – [x – [x])}



$\Rightarrow$ always continuous in [–2, 2]

but non differentiable at 7 Points

8. Official Ans. by NTA (3)

Sol. $f(x) = |(x - 3)(x + 1)| \cdot e^{(3x-2)^2}$

$$f(x) = \begin{cases} (x-3)(x+1).e^{(3x-2)^2} & ; \ x \in (3, \infty) \\ -(x-3)(x+1).e^{(3x-2)^2} & ; \ x \in [-1, 3] \\ (x-3).(x+1).e^{(3x-2)^2} & ; \ x \in (-\infty, -1) \end{cases}$$

Clearly, non-differentiable at x = –1 & x = 3.

9. Official Ans. by NTA (2)

Sol. $f(x) = |2x+1| - 3|x+2| + |x^2 + x - 2|$

$= |2x+1| - 3|x+2| + |x+2||x-1|$

$= |2x+1| + |x+2|(|x-1| - 3)$

Critical points are $x = \frac{-1}{2}, -2, -1$

but x = –2 is making a zero.

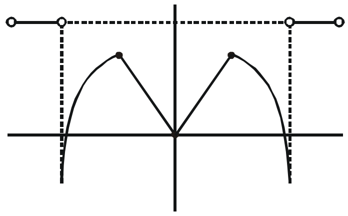
twice in product so, points of non differentability are $\boxed{x = \frac{-1}{2} \text{ and } x = -1}$

$\therefore$ Number of points of non-differentiability = $\boxed{2}$

10. Official Ans. by NTA (5)

Sol. $f(x)=\begin{cases}\min\{|x|,2-x^2\} & , -2\le x\le 2\\ [|x|] & , 2<|x|\le 3\end{cases}$

$\Rightarrow x \in [-3,-2)\cup(2,3]$



Number of points of non-differentiability in $(-3,3)=5$

11. Official Ans. by NTA (2)

Sol. $f(g(x))=\begin{cases}g(x)+2, & g(x)<0\\ (g(x))^2, & g(x)\ge 0\end{cases}$

$=\begin{cases}x^3+2, & x<0\\ x^6, & x\in[0,1)\\ (3x-2)^2, & x\in[1,\infty)\end{cases}$

$(f o g(x))'=\begin{cases}3x^2, & x<0\\ 6x^5, & x\in(0,1)\\ 2(3x-2)\times 3, & x\in(1,\infty)\end{cases}$

At 'O'

L.H.L. $\ne$ R.H.L. (Discontinuous)

At '1'

L.H.D. = 6 = R.H.D.

$\Rightarrow$ $f o g(x)$ is differentiable for $x\in\mathbb{R}-\{0\}$

12. Official Ans. by NTA (4)

Sol. $f(x)=\begin{cases}\frac{1}{|x|}, & |x|\ge 1\\ ax^2+b, & |x|<1\end{cases}$

at x = 1 function must be continuous

So, 1 = a + b ...(1)

differentiability at x = 1

$\left(-\frac{1}{x^2}\right)_{x=1}=(2ax)_{x=1}$

$\Rightarrow -1=2a\Rightarrow a=-\frac{1}{2}$

$(1)\Rightarrow b=1+\frac{1}{2}=\frac{3}{2}$

13. Official Ans. by NTA (3)

Sol. If $f(x+y)=f(x).f(y)$ & $f'(0)=3$ then

$f(x)=a^x\Rightarrow f'(x)=a^x.\ell na$

$\Rightarrow f'(0)=\ell na=3\Rightarrow a=e^3$

$\Rightarrow f(x)=(e^3)^x=e^{3x}$

$\lim\limits_{x\to 0}\frac{f(x)-1}{x}=\lim\limits_{x\to 0}\left(\frac{e^{3x}-1}{3x}\times 3\right)=1\times 3=3$`