DEFINITE INTEGRATION

1. Let a be a positive real number such that $\int_0^a e^{x-[x]}dx = 10e - 9$ where [x] is the greatest integer less than or equal to x. Then a is equal to :

(1) $10 - \log_e(1+e)$ (2) $10 + \log_e 2$

(3) $10 + \log_e 3$ (4) $10 + \log_e(1+e)$

2. The value of the integral $\int_{-1}^{1} \log_e(\sqrt{1-x}+\sqrt{1+x})dx$ is equal to :

(1) $\frac{1}{2}\log_e 2 + \frac{\pi}{4} - \frac{3}{2}$ (2) $2\log_e 2 + \frac{\pi}{4} - 1$

(3) $\log_e 2 + \frac{\pi}{2} - 1$ (4) $2\log_e 2 + \frac{\pi}{2} - \frac{1}{2}$

3. Let $g(t) = \int_{-\pi/2}^{\pi/2} \cos\left(\frac{\pi}{4}t + f(x)\right)dx$, where $f(x) = \log_e\left(x+\sqrt{x^2+1}\right)$, $x \in \mathbf{R}$. Then which one of the following is correct ?

(1) $g(1) = g(0)$ (2) $\sqrt{2}g(1) = g(0)$

(3) $g(1) = \sqrt{2}g(0)$ (4) $g(1) + g(0) = 0$

4. If $\int_0^{100\pi} \frac{\sin^2 x}{e^{\left(\frac{x}{\pi}-\left[\frac{x}{\pi}\right]\right)}}dx = \frac{\alpha\pi^3}{1+4\pi^2}, \alpha \in \mathbf{R}$ where [x] is the greatest integer less than or equal to x, then the value of α is :

(1) $200(1-e^{-1})$ (2) $100(1-e)$

(3) $50(e-1)$ (4) $150(e^{-1}-1)$

5. The value of the definite integral $\int_{\pi/24}^{5\pi/24} \frac{dx}{1+\sqrt[3]{\tan 2x}}$ is :

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{12}$ (4) $\frac{\pi}{18}$

6. Let $f : [0, \infty) \to [0, \infty)$ be defined as $f(x) = \int_0^x [y]dy$ where [x] is the greatest integer less than or equal to x. Which of the following is true?

(1) f is continuous at every point in $[0, \infty)$ and differentiable except at the integer points.

(2) f is both continuous and differentiable except at the integer points in $[0, \infty)$.

(3) f is continuous everywhere except at the integer points in $[0, \infty)$.

(4) f is differentiable at every point in $[0, \infty)$.

7. If $f(x) = \begin{cases} \int_0^x (5+|1-t|)dt, & x > 2 \\ 5x+1, & x \le 2 \end{cases}$, then

(1) f(x) is not continuous at x = 2

(2) f(x) is everywhere differentiable

(3) f(x) is continuous but not differentiable at x = 2

(4) f(x) is not differentiable at x = 1

8. The value of the integral $\int_{-1}^{1} \log\left(x+\sqrt{x^2+1}\right)dx$ is:

(1) 2 (2) 0 (3) –1 (4) 1

9. The value of the definite integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+e^{x\cos x})(\sin^4 x+\cos^4 x)}$ is equal to :

(1) $-\frac{\pi}{2}$ (2) $\frac{\pi}{2\sqrt{2}}$ (3) $-\frac{\pi}{4}$ (4) $\frac{\pi}{\sqrt{2}}$

10. Let the domain of the function $f(x) = \log_4\left(\log_5\left(\log_3(18x-x^2-77)\right)\right)$ be (a, b). Then the value of the integral $\int_a^b \frac{\sin^3 x}{(\sin^3 x+\sin^3(a+b-x))}dx$ is equal to ______.

11. Let f : (a,b) $\rightarrow$ **R** be twice differentiable function such that $f(x) = \int_a^x g(t)\,dt$ for a differentiable function g(x). If $f(x) = 0$ has exactly five distinct roots in (a, b), then g(x)g'(x) = 0 has at least :

(1) twelve roots in (a, b) (2) five roots in (a, b)

(3) seven roots in (a, b) (4) three roots in (a, b)

12. If $\int_0^{\pi}(\sin^3 x)e^{-\sin^2 x}\,dx = \alpha - \frac{\beta}{e}\int_0^1 \sqrt{t}\,e^t dt$, then $\alpha + \beta$ is equal to _____ .

13. The value of $\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\left(\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2 - 2\right)^{1/2} dx$ is :

(1) $\log_e 4$ (2) $\log_e 16$

(3) $2\log_e 16$ (4) $4\log_e(3+2\sqrt{2})$

14. The value of $\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{2n-1}\frac{n^2}{n^2+4r^2}$ is :

(1) $\frac{1}{2}\tan^{-1}(2)$ (2) $\frac{1}{2}\tan^{-1}(4)$

(3) $\tan^{-1}(4)$ (4) $\frac{1}{4}\tan^{-1}(4)$

15. If the value of the integral $\int_0^5 \frac{x+[x]}{e^{x-[x]}}dx = \alpha e^{-1} + \beta$, where $\alpha, \beta \in$ **R**, $5\alpha + 6\beta = 0$, and [x] denotes the greatest integer less than or equal to x; then the value of $(\alpha + \beta)^2$ is equal to :

(1) 100 (2) 25 (3) 16 (4) 36

16. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1+\sin^2 x}{1+\pi^{\sin x}}\right)dx$ is

(1) $\frac{\pi}{2}$ (2) $\frac{5\pi}{4}$ (3) $\frac{3\pi}{4}$ (4) $\frac{3\pi}{2}$

17. If $U_n = \left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2 \ldots\left(1+\frac{n^2}{n^2}\right)^n$, then $\lim_{n\to\infty}(U_n)^{\frac{-4}{n^2}}$ is equal to :

(1) $\frac{e^2}{16}$ (2) $\frac{4}{e}$ (3) $\frac{16}{e^2}$ (4) $\frac{4}{e^2}$

18. $\int_6^{16}\frac{\log_e x^2}{\log_e x^2 + \log_e(x^2-44x+484)}dx$ is equal to :

(1) 6 (2) 8 (3) 5 (4) 10

19. The value of the integral $\int_0^1 \frac{\sqrt{x}\,dx}{(1+x)(1+3x)(3+x)}$ is :

(1) $\frac{\pi}{8}\left(1-\frac{\sqrt{3}}{2}\right)$ (2) $\frac{\pi}{4}\left(1-\frac{\sqrt{3}}{6}\right)$

(3) $\frac{\pi}{8}\left(1-\frac{\sqrt{3}}{6}\right)$ (4) $\frac{\pi}{4}\left(1-\frac{\sqrt{3}}{2}\right)$

20. Let [t] denote the greatest integer $\leq$ t. Then the value of $8\cdot\int_{-\frac{1}{2}}^{1}([2x]+|x|)\,dx$ is _________.

21. If $x\,\phi(x) = \int_5^x (3t^2 - 2\phi'(t))\,dt$, $x > -2$, and $\phi(0) = 4$, then $\phi(2)$ is _________.

22. If [x] is the greatest integer $\leq$ x, then $\pi^2\int_0^2\left(\sin\frac{\pi x}{2}\right)(x-[x])^{[x]}dx$ is equal to :

(1) $2(\pi - 1)$ (2) $4(\pi - 1)$

(3) $4(\pi + 1)$ (4) $2(\pi + 1)$

23. The function f(x), that satisfies the condition $f(x) = x + \int_0^{\pi/2}\sin x\cdot\cos y\, f(y)\,dy$, is :

(1) $x + \frac{2}{3}(\pi-2)\sin x$ (2) $x + (\pi + 2)\sin x$

(3) $x + \frac{\pi}{2}\sin x$ (4) $x + (\pi - 2)\sin x$

24. The value of the integral, $\int_1^3 [x^2-2x-2]\,dx$, where [x] denotes the greatest integer less than or equal to x, is :

(1) $-\sqrt{2}-\sqrt{3}+1$ (2) $-\sqrt{2}-\sqrt{3}-1$

(3) –5 (4) –4

25. Let f(x) be a differentiable function defined on [0, 2] such that f '(x) = f '(2 – x) for all x ∈ (0, 2), f(0) = 1 and f(2) = e^2. Then the value of $\int_0^2 f(x)dx$ is :

(1) $1-e^2$ (2) $1+e^2$

(3) $2(1-e^2)$ (4) $2(1+e^2)$

26. If $\int_{-a}^{a}(|x|+|x-2|)dx = 22, (a>2)$ and [x] denotes the greatest integer ≤ x, then $\int_{a}^{-a}(x+[x])\,dx$ is equal to _______.

27. The value of $\int_{-1}^{1} x^2 e^{[x^3]}dx$, where [t] denotes the greatest integer ≤ t, is :

(1) $\frac{e-1}{3e}$ (2) $\frac{e+1}{3}$ (3) $\frac{e+1}{3e}$ (4) $\frac{1}{3e}$

28. $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+........+\frac{n}{(2n-1)^2}\right]$ is equal to :

(1) $\frac{1}{2}$ (2) 1 (3) $\frac{1}{3}$ (4) $\frac{1}{4}$

29. The value of $\int_{-2}^{2}|3x^2-3x-6|\,dx$ is ______.

30. For x > 0, if $f(x)=\int_1^x \frac{\log_e t}{(1+t)}dt$, then $f(e)+f\left(\frac{1}{e}\right)$ is equal to :

(1) 1 (2) –1 (3) $\frac{1}{2}$ (4) 0

31. Let $f(x)=\int_0^x e^t f(t)dt + e^x$ be a differentiable function for all x ∈ R. Then $f(x)$ equals :

(1) $2e^{(e^x-1)}-1$ (2) $e^{e^x}-1$

(3) $2e^{e^x}-1$ (4) $e^{(e^x-1)}$

32. If $I_{m,n}=\int_0^1 x^{m-1}(1-x)^{n-1}dx$, for m, n ≥ 1 and $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx = \alpha\, I_{m,n}$, $\alpha \in R$, then α equals ____.

33. The value of the integral $\int_0^{\pi}|\sin 2x|\,dx$ is

34. The value of $\int_{-\pi/2}^{\pi/2}\frac{\cos^2 x}{1+3^x}dx$ is

(1) $\frac{\pi}{4}$ (2) 4π (3) $\frac{\pi}{2}$ (4) 2π

35. If $I_n=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cot^n x\,dx$, then :

(1) $\frac{1}{I_2+I_4},\frac{1}{I_3+I_5},\frac{1}{I_4+I_6}$ are in G.P.

(2) $I_2+I_4, I_3+I_5, I_4+I_6$ are in A.P.

(3) $I_2+I_4, (I_3+I_5)^2, I_4+I_6$ are in G.P.

(4) $\frac{1}{I_2+I_4},\frac{1}{I_3+I_5},\frac{1}{I_4+I_6}$ are in A.P.

36. The value of $\sum_{n=1}^{100}\int_{n-1}^{n} e^{x-[x]}dx$, where [x] is the greatest integer ≤ x, is

(1) 100(e – 1) (2) 100(1 – e)

(3) 100e (4) 100 (1 + e)

37. Consider the integral

$$I=\int_0^{10}\frac{[x]\,e^{[x]}}{e^{x-1}}dx,$$

where [x] denotes the greatest integer less than or equal to x. Then the value of I is equal to:

(1) 9(e – 1) (2) 45(e + 1)

(3) 45(e – 1) (4) 9(e + 1)

38. Let $P(x) = x^2 + bx + c$ be a quadratic polynomial with real coefficients such that $\int_0^1 P(x)dx = 1$ and P(x) leaves remainder 5 when it is divided by (x – 2). Then the value of 9(b + c) is equal to:

(1) 9 (2) 15 (3) 7 (4) 11

39. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) + f(x + 1) = 2$, for all $x \in \mathbb{R}$. If $I_1 = \int_0^8 f(x)dx$ and $I_2 = \int_{-1}^3 f(x)dx$, then the value of $I_1 + 2I_2$ is equal to ________.

40. Let $f : R \to R$ be defined as $f(x) = e^{-x}\sin x$. If $F : [0, 1] \to R$ is a differentiable function such that $F(x) = \int_0^x f(t)\,dt$, then the value of $\int_0^1 (F'(x) + f(x))e^x dx$ lies in the interval

(1) $\left[\frac{327}{360}, \frac{329}{360}\right]$ (2) $\left[\frac{330}{360}, \frac{331}{360}\right]$

(3) $\left[\frac{331}{360}, \frac{334}{360}\right]$ (4) $\left[\frac{335}{360}, \frac{336}{360}\right]$

41. If the integral $\int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}}dx = \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$, where α, β, γ are integers and [x] denotes the greatest integer less than or equal to x, then the value of $\alpha + \beta + \gamma$ is equal to :

(1) 0 (2) 20 (3) 25 (4) 10

42. Let $I_n = \int_1^e x^{19}(\log|x|)^n dx$, where $n \in N$. If $(20)I_{10} = \alpha I_9 + \beta I_8$, for natural numbers α and β, then $\alpha - \beta$ equal to _______.

43. Which of the following statements is incorrect for the function $g(\alpha)$ for $\alpha \in R$ such that

$$g(\alpha) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^\alpha x}{\cos^\alpha x + \sin^\alpha x}dx$$

(1) $g(\alpha)$ is a strictly increasing function

(2) $g(\alpha)$ has an inflection point at $\alpha = -\frac{1}{2}$

(3) $g(\alpha)$ is a strictly decreasing function

(4) $g(\alpha)$ is an even function

44. Let $f(x)$ and $g(x)$ be two functions satisfying $f(x^2) + g(4 - x) = 4x^3$ and $g(4 - x) + g(x) = 0$, then the value of $\int_{-4}^4 f(x)^2 dx$ is

45. Let $g(x) = \int_0^x f(t)\,dt$, where f is continuous function in [0, 3] such that $\frac{1}{3} \le f(t) \le 1$ for all $t \in [0, 1]$ and $0 \le f(t) \le \frac{1}{2}$ for all $t \in (1, 3]$. The largest possible interval in which g(3) lies is :

(1) $\left[-1, -\frac{1}{2}\right]$ (2) $\left[-\frac{3}{2}, -1\right]$

(3) $\left[\frac{1}{3}, 2\right]$ (4) [1, 3]

SOLUTION

1. Official Ans. by NTA (2)

Sol. $a > 0$

Let $n \le a < n+1, n \in W$

$\therefore\ a = [a] + \{a\}$

$\Downarrow \quad \Downarrow$

G.I.F Fractional part

Here $[a] = n$

Now, $\int_0^a e^{x-[x]}\,dx = 10e - 9$

$\Rightarrow \int_0^n e^{\{x\}}\,dx + \int_n^a e^{x-[x]}\,dx = 10e - 9$

$\therefore\ n\int_0^1 e^x\,dx + \int_n^a e^{x-n}\,dx = 10e - 9$

$\Rightarrow n(e-1) + (e^{a-n} - 1) = 10e - 9$

$\therefore\ \boxed{n=0}$ and $\{a\} = \log_e 2$

So, $a = [a] + \{a\} = (10 + \log_e 2)$

$\Rightarrow$ Option (2) is correct.

2. Official Ans. by NTA (2)

ALLEN Ans. (3)

Sol. Let $I = 2\int_0^1 \underbrace{\ln\left(\sqrt{1-x}+\sqrt{1+x}\right)}_{(I)} \underbrace{1}_{(II)}\,dx$

(I.B.P.)

$\therefore\ I = 2\left[\left(x.\ln\left(\sqrt{1-x}+\sqrt{1-x}\right)\right)_0^1\right.$

$\left.-\int_0^1 x.\left(\frac{1}{\sqrt{1-x}+\sqrt{1+x}}\right)\cdot\left(\frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}}\right)dx\right]$

$= 2\left(\ln\sqrt{2} - 0\right) - \frac{2}{2}\int_0^1 \frac{x\sqrt{1-x} - \sqrt{1+x}\ dx}{\left(\sqrt{1-x}+\sqrt{1+x}\right)\sqrt{1-x^2}}$

$= \left(\log_e 2\right) - \int_0^1 \frac{x\cdot\left(2 - 2\sqrt{1-x^2}\right)}{-2x\sqrt{1-x^2}}\,dx$ (After rationalisation)

$= \left(\log_e 2\right) + \int_0^1 \left(\frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)dx$

$= \left(\log_e 2\right) + \left(\sin^{-1} x\right)_0^1 - 1$

$= \log_e 2 + \left(\frac{\pi}{2} - 0\right) - 1$

$\therefore\ I = \left(\log_e 2\right) + \frac{\pi}{2} - 1$

$\Rightarrow$ Option (3) is correct.

3. Official Ans. by NTA (2)

Sol. $g(t) = \int_{-\pi/2}^{\pi/2}\left(\cos\frac{\pi}{4}t + f(x)\right)dx$

$g(t) = \pi\cos\frac{\pi}{4}t + \int_{-\pi/2}^{\pi/2} f(x)\,dx$

$g(t) = \pi\cos\frac{\pi}{4}t$

$g(1) = \frac{\pi}{\sqrt{2}}, g(0) = \pi$

4. Official Ans. by NTA (1)

Sol. $I = \int_0^{100\pi} \frac{\sin^2 x}{e^{\{x/\pi\}}} dx = 100\int_0^{\pi} \frac{\sin^2 x}{e^{x/\pi}} dx$

$100\int_0^{\pi} e^{-x/\pi} \frac{(1-\cos 2x)}{2} dx$

$= 50\left\{\int_0^{\pi} e^{-x/\pi} dx - \int_0^{\pi} e^{-x/\pi} \cos 2x\, dx\right\}$

$I_1 = \int_0^{\pi} e^{-x/\pi} dx = \left[-\pi e^{-x/\pi}\right]_0^{\pi} = \pi(1-e^{-1})$

$I_2 = \int_0^{\pi} e^{-x/\pi} \cos 2x\, dx$

$= -\pi e^{-x/\pi} \cos 2x\Big]_0^{\pi} - \int -\pi e^{-x/\pi}(-2\sin 2x) dx$

$= \pi(1-e^{-1}) - 2\pi\int_0^{\pi} e^{-x/\pi} \sin 2x\, dx$

$= \pi(1-e^{-1}) - 2\pi\left\{-\pi e^{-x/\pi}\sin 2x\Big]_0^{\pi} - \int_0^{\pi} -\pi e^{-x/\pi} 2\cos 2x dx\right\}$

$= \pi(1-e^{-1}) - 4\pi^2 I_2$

$\Rightarrow I_2 = \frac{\pi(1-e^{-1})}{1+4\pi^2}$

$\therefore I = 50\left\{\pi(1-e^{-1}) - \frac{\pi(1-e^{-1})}{1+4\pi^2}\right\}$

$= \frac{200(1-e^{-1})\pi^3}{1+4\pi^2}$

5. Official Ans. by NTA (3)

Sol. Let $I = \int_{\pi/24}^{5\pi/24} \frac{(\cos 2x)^{1/3}}{(\cos 2x)^{1/3} + (\sin 2x)^{1/3}} dx$ ....(i)

$\Rightarrow I = \int_{\pi/24}^{5\pi/24} \frac{\left(\cos\left\{2\left(\frac{\pi}{4}-x\right)\right\}\right)^{\frac{1}{3}}}{\left(\cos\left\{2\left(\frac{\pi}{4}-x\right)\right\}\right)^{\frac{1}{3}} + \left(\sin\left\{2\left(\frac{\pi}{4}-x\right)\right\}\right)^{\frac{1}{3}}} dx$

$\left\{\int_a^b f(x)\, dx = \int_a^b f(a+b-x) dx\right\}$

So $I = \int_{\pi/24}^{5\pi/24} \frac{(\sin 2x)^{1/3}}{(\sin 2x)^{1/3} + (\cos 2x)^{1/3}} dx$ ....(ii)

Hence $2I = \int_{\pi/24}^{5\pi/24} dx$ [(i) + (ii)]

$\Rightarrow 2I = \frac{4\pi}{24} \Rightarrow \boxed{I = \frac{\pi}{12}}$

6. Official Ans. by NTA (1)

Sol. $f : [0, \infty) \to [0, \infty)$, $f(x) = \int_0^x [y]\, dy$

Let $x = n + f$, $f \in (0, 1)$

So $f(x) = 0 + 1 + 2 + \ldots + (n-1) + \int_n^{n+f} n\, dy$

$f(x) = \frac{n(n-1)}{2} + nf$

$= \frac{[x]([x]-1)}{2} + [x]\{x\}$

Note $\lim_{x\to n^+} f(x) = \frac{n(n-1)}{2}$,

$\lim_{x\to n^-} f(x) = \frac{(n-1)(n-2)}{2} + (n-1) = \frac{n(n-1)}{2}$

$f(x) = \frac{n(n-1)}{2}$ $(n \in N_0)$

so $f(x)$ is cont. $\forall$ $x \geq 0$ and diff. except at integer points

7. Official Ans. by NTA (3)

Sol. $f(x) = \int_0^1 (5+(1-t))\,dt + \int_1^x (5+(t-1))\,dt$

$= 6 - \frac{1}{2} + \left(4t + \frac{t^2}{2}\right)\Big|_1^x$

$= \frac{11}{2} + 4x + \frac{x^2}{2} - 4 - \frac{1}{2}$

$= \frac{x^2}{2} + 4x + 1$

$f(2^+) = 2 + 8 + 1 = 11$

$f(2) = f(2^-) = 5 \times 2 + 1 = 11$

$\Rightarrow$ continuous at x = 2

Clearly differentiable at x = 1

Lf'(2) = 5

Rf'(2) = 6

$\Rightarrow$ not differentiable at x = 2

8. Official Ans. by NTA (2)

Sol. Let $I = \int_{-1}^{1} \log\left(x + \sqrt{x^2+1}\right) dx$

$\because \log\left(x+\sqrt{x^2+1}\right)$ is an odd function

$\therefore I = 0$

9. Official Ans. by NTA (2)

Sol. $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+e^{x\cos x})(\sin^4 x + \cos^4 x)}$ ...(1)

Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$I = \int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{-x\cos x})(\sin^4 x + \cos^4 x)}$

Add (1) and (2)

$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\sin^4 x + \cos^4 x}$

$2I = 2\int_0^{\frac{\pi}{4}} \frac{dx}{\sin^4 x + \cos^4 x}$

$I = \int_0^{\frac{\pi}{4}} \frac{(1+\tan^2 x)\sec^2 x}{\tan^4 x + 1} dx$

$I = \int_0^{\frac{\pi}{4}} \frac{\left(1+\frac{1}{\tan^2 x}\right)\sec^2 x}{\left(\tan x - \frac{1}{\tan x}\right)^2 + 2} dx$

$\tan x - \frac{1}{\tan x} = t$

$\left(1 + \frac{1}{\tan^2 x}\right)\sec^2 x dx = dt$

$I = \int_{-\infty}^{0} \frac{dt}{t^2+2} = \left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_{-\infty}^{0}$

$I = 0 - \frac{1}{\sqrt{2}}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2\sqrt{2}}$

10. Official Ans. by NTA (1)

Sol. For domain

$\log_5\left(\log_3\left(18x - x^2 - 77\right)\right) > 0$

$\log_3\left(18x - x^2 - 77\right) > 1$

$18x - x^2 - 77 > 3$

$x^2 - 18x + 80 < 0$

$x \in (8, 10)$

$\Rightarrow a = 8$ and b = 10

$I = \int_a^b \frac{\sin^3 x}{\sin^3 x + \sin^3(a+b-x)} dx$

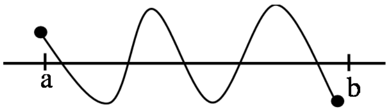
$I = \int_a^b \frac{\sin^3(a+b-x)}{\sin^3 x + \sin^3(a+b-x)}$

$2I = (b-a) \Rightarrow I = \frac{b-a}{2}$ $(\because a = 8 \text{ and } b = 10)$

$I = \frac{10-8}{2} = 1$

11. Official Ans. by NTA (3)

Sol.



$$f(x)=\int_a^x g(t)dt$$

$f(x)\rightarrow 5$

$f'(x)\rightarrow 4$

$g(x)\rightarrow 4$

$g'(x)\rightarrow 3$

12. Official Ans. by NTA (5)

Sol. $I=2\int_0^{\pi/2}\sin^3 x\, e^{-\sin^2 x}dx$

$$=2\int_0^{\pi/2}\sin x\, e^{-\sin^2 x}dx+$$

$$\int_0^{\pi/2}\underbrace{\cos x}_{I}\ \underbrace{e^{-\sin^2 x}(-\sin 2x)}_{II}\,dx$$

$$=2\int_0^{\pi/2}\sin x e^{-\sin^2 x}dx+\left[\cos x\, e^{-\sin^2 x}\right]_0^{\pi/2}$$

$$+\int_0^{\pi/2}\sin x\, e^{-\sin^2 x}\,dx$$

$$=3\int_0^{\pi/2}\sin x\, e^{-\sin^2 x}dx-1$$

$$=\frac{3}{2}\int_{-1}^{0}\frac{e^{\alpha}d\alpha}{\sqrt{1+\alpha}}-1\ \ (\text{Put } -\sin^2 x=t)$$

$$=\frac{3}{2e}\int_0^1\frac{e^x}{\sqrt{x}}dx-1\ \ (\text{put } 1+\alpha=x)$$

$$=\frac{3}{2e}\int_0^1\underbrace{e^x}_{II}\underbrace{\frac{1}{\sqrt{x}}}_{II}dx-1$$

$$=2-\frac{3}{e}\int_0^1 e^x\sqrt{x}\,dx$$

Hence, $\alpha+\beta=\boxed{5}$

13. Official Ans. by NTA (2)

Sol. $I=\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\left(\left(\frac{x+1}{x-1}-\frac{x-1}{x+1}\right)^2\right)^{1/2}dx$

$$I=\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\left|\frac{4x}{x^2-1}\right|dx\Rightarrow I=2.4\int_0^{1/\sqrt{2}}\left|\frac{x}{x^2-1}\right|dx$$

$$\Rightarrow I=-4\int_0^{1/\sqrt{2}}\frac{2x}{x^2-1}dx\Rightarrow I=-4\ln\left|x^2-1\right|_0^{1/\sqrt{2}}$$

$\Rightarrow I = 4 \ln 2 \Rightarrow I = \ln 16$

14. Official Ans. by NTA (2)

Sol. $L=\lim_{n\to\infty}\frac{1}{n}.\sum_{r=0}^{2n-1}\frac{1}{1+4\left(\frac{r}{n}\right)^2}$

$$\Rightarrow L=\int_0^2\frac{1}{1+4x^2}dx$$

$$\Rightarrow L=\frac{1}{2}\tan^{-1}(2x)\Big|_0^2\Rightarrow L=\frac{1}{2}\tan^{-1}4$$

15. Official Ans. by NTA (2)

Sol. $I=\int_0^5\frac{x+[x]}{e^{x-[x]}}dx$

$$\int_0^1\frac{x}{e^x}dx+\int_1^2\frac{x+1}{e^{x-1}}dx+\int_2^3\frac{x+2}{e^{x-2}}dx+.....\int_4^5\frac{x+4}{e^{x-4}}dx$$

$\Downarrow$ $\Downarrow$ $\Downarrow$

x = t + 1 x = z + 2 x = y + 4

$$\int_0^1\frac{t+2}{e^t}dt+\int_0^1\frac{z+4}{e^z}dz+.....+\int_0^1\frac{y+8}{e^y}dx$$

$$\Rightarrow\int_0^5\frac{5x+20}{e^x}dt=5\int_0^1\frac{x+4}{e^x}dx$$

$$\Rightarrow\ 5\int_0^1(x+4)e^{-x}dx$$

$$\Rightarrow\ 5e^{-x}(-x-5)\big|_0^1\Rightarrow-\frac{30}{e}+25$$

$\alpha = -30$

$\beta = 25 \Rightarrow 5\alpha + 6\beta = 0$

$(\alpha + \beta)^2 = 5^2 = 25$

16. Official Ans. by NTA (3)

Sol. $I=\int_0^{\pi/2}\frac{(1+\sin^2 x)}{(1+\pi^{\sin x})}+\frac{\pi^{\sin x}(1+\sin^2 x)}{(1+\pi^{\sin x})}dx$

$I=\int_0^{\pi/2}(1+\sin^2 x)dx$

$I=\frac{\pi}{2}+\frac{\pi}{2}\cdot\frac{1}{2}=\frac{3\pi}{4}$

17. Official Ans. by NTA (1)

Sol. $U_n=\prod_{r=1}^{n}\left(1+\frac{r^2}{n^2}\right)^r$

$L=\lim_{n\to\infty}(U_n)^{-4/n^2}$

$\log L=\lim_{n\to\infty}\frac{-4}{n^2}\sum_{r=1}^{n}\log\left(1+\frac{r^2}{n^2}\right)^r$

$\Rightarrow \log L=\lim_{n\to\infty}\sum_{r=1}^{n}-\frac{4r}{n}\cdot\frac{1}{n}\log\left(1+\frac{r^2}{n^2}\right)$

$\Rightarrow \log L\Rightarrow -4\int_0^1 x\log(1+x^2)dx$

put $1+x^2=t$

Now, $2xdx=dt$

$=-2\int_1^2\log(t)dt=-2[t\log t-t]_1^2$

$\Rightarrow \log L=-2(2\log 2-1)$

$\therefore\ L=e^{-2(2\log 2-1)}$

$=e^{-2\left(\log\left(\frac{4}{e}\right)\right)}$

$=e^{\log\left(\frac{4}{e}\right)^{-2}}$

$=\left(\frac{e}{4}\right)^2=\frac{e^2}{16}$

18. Official Ans. by NTA (3)

Sol. Let $I=\int_6^{16}\frac{\log_e x^2}{\log_e x^2+\log_e(x^2-44x+484)}dx$

$I=\int_6^{16}\frac{\log_e x^2}{\log_e x^2+\log_e(x-22)^2}dx$ ...(1)

We know

$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$ (king)

So

$I=\int_6^{16}\frac{\log_e(22-x)^2}{\log_e(22-x)^2+\log_e(22-(22-x))^2}$

$I=\int_0^{16}\frac{\log_e(22-x)^2}{\log_e x^2+\log_e(22-x)^2}dx$ ...(2)

(1) + (2)

$2I=\int_6^{16}1.dx=10$

I = 5

19. Official Ans. by NTA (1)

Sol. $I=\int_0^1 \frac{\sqrt{x}}{(1+x)(1+3x)(3+x)}dx$

Let $x=t^2 \Rightarrow dx=2t.dt$

$$I=\int_0^1 \frac{t(2t)}{(t^2+1)(1+3t^2)(3+t^2)}dt$$

$$I=\int_0^1 \frac{(3t^2+1)-(t^2+1)}{(3t^2+1)(t^2+1)(3+t^2)}dt$$

$$I=\int_0^1 \frac{dt}{(t^2+1)(3+t^2)}-\int_0^1 \frac{dt}{(1+3t^2)(3+t^2)}$$

$$=\frac{1}{2}\int_0^1 \frac{(3+t^2)-(t^2+1)}{(t^2+1)(3+t^2)}dt+\frac{1}{8}\int_0^1 \frac{(1+3t^2)-3(3+t^2)}{(1+3t^2)(3+t^2)}dt$$

$$=\frac{1}{2}\int_0^1 \frac{dt}{1+t^2}-\frac{1}{2}\int_0^1 \frac{dt}{t^2+3}$$

$$+\frac{1}{8}\int_0^1 \frac{dt}{t^2+3}-\frac{3}{8}\int_0^1 \frac{dt}{(1+3t^2)}$$

$$=\frac{1}{2}\int_0^1 \frac{dt}{t^2+1}-\frac{3}{8}\int_0^1 \frac{dt}{t^2+3}-\frac{3}{8}\int_0^1 \frac{dt}{1+3t^2}$$

$$=\frac{1}{2}\left(\tan^{-1}(t)\right)_0^1-\frac{3}{8\sqrt{3}}\left(\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right)_0^1$$

$$-\frac{3}{8\sqrt{3}}\left(\tan^{-1}\left(\sqrt{3}t\right)\right)_0^1$$

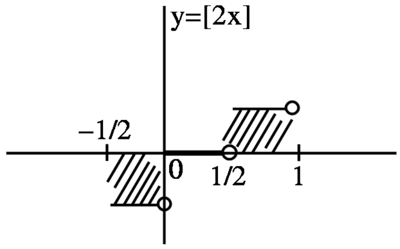
$$=\frac{1}{2}\left(\frac{\pi}{4}\right)-\frac{\sqrt{3}}{8}\left(\frac{\pi}{6}\right)-\frac{\sqrt{3}}{8}\left(\frac{\pi}{3}\right)$$

$$=\frac{\pi}{8}-\frac{\sqrt{3}}{16}\pi$$

$$=\frac{\pi}{8}\left(1-\frac{\sqrt{3}}{2}\right)$$

20. Official Ans. by NTA (5)

Sol. $I=\int_{-1/2}^{1}\left([2x]+|x|\right)dx$



$$=\int_{-1/2}^{1}[2x]\,dx+\int_{-1/2}^{1}|x|\,dx$$

$$=0+\int_{-1/2}^{0}(-x)\,dx+\int_0^1 x\,dx$$

$$=\left(-\frac{x^2}{2}\right)_{-1/2}^{0}+\left(\frac{x^2}{2}\right)_0^1$$

$$=\left(0+\frac{1}{8}\right)+\frac{1}{2}$$

$$=\frac{5}{8}$$

$8I=5$

21. Official Ans. by NTA (4)

Sol. $x\phi(x)=\int_5^x 3t^2-2\phi'(t)\,dt$

$x\phi(x)=x^3-125-2[\phi(x)-\phi(5)]$

$x\phi(x)=x^3-125-2\phi(x)-2\phi(5)$

$\phi(0)=4 \Rightarrow \phi(5)=-\frac{133}{2}$

$\phi(x)=\frac{x^3+8}{x+2}$

$\phi(2)=4$

22. Official Ans. by NTA (2)

Sol. $\pi^2\left[\int_0^1 \sin\frac{\pi x}{2}dx+\int_1^2 \sin\frac{\pi x}{2}(x-1)\,dx\right]$

$$=\pi^2\left[-\frac{2}{\pi}\left(\cos\frac{\pi x}{2}\right)+\left((x-1)\left(-\frac{2}{\pi}\cos\frac{\pi x}{2}\right)\right)_1^2-\int_1^2-\frac{2}{\pi}\cos\frac{\pi x}{2}dx\right]$$

$$=\pi^2\left[0+\frac{2}{\pi}+\frac{2}{\pi}+\frac{2}{\pi}.\frac{2}{\pi}\left(\sin\frac{\pi x}{2}\right)_1^2\right]$$

$=4\pi-4=4(\pi-1)$

23. Official Ans. by NTA (4)

Sol. $f(x) = x + \int_0^{\pi/2} \sin x \cos y f(y) dy$

$f(x) = x + \sin x \underbrace{\int_0^{\pi/2} \cos y f(y) dy}_{K}$

$\Rightarrow f(x) = x + K \sin x$

$\Rightarrow f(y) = y + K \sin y$

Now $K = \int_0^{\pi/2} \cos y (y + K \sin y) dy$

$K = \underset{\text{Apply IBP}}{\int_0^{\pi/2} y \cos dy} + \underset{\text{Put } \sin y = t}{\int_0^{\pi/2} \cos y \sin y dy}$

$K = (y \sin y)_0^{\pi/2} - \int_0^{\pi/2} \sin dy + K\int_0^1 t\, dt$

$\Rightarrow K = \frac{\pi}{2} - 1 + K\left(\frac{1}{2}\right)$

$\Rightarrow K = \pi - 2$

So $f(x) = x + (\pi - 2) \sin x$

Option (4)

24. Official Ans. by NTA (2)

Sol. $\int_1^3 \left(\left[(x-1)^2\right] - 3\right) dx$

$= \int_1^2 [x^2] - 3\int_1^3 dx$

$= \int_1^3 0\, dx + \int_1^{\sqrt{2}} 1.dx + \int_{\sqrt{2}}^{\sqrt{3}} 2.dx + \int_{\sqrt{3}}^{2} 3.dx - 6$

$= \sqrt{2} - 1 + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3}) - 6$

$= -\sqrt{2} - \sqrt{3} - 1$

25. Official Ans. by NTA (2)

Sol. $f'(x) = f'(2-x)$

$f(x) = -f(2-x) + c$

put $x = 0$

$f'(0) = -f'(2) + c$

$c = f(0) + f(2) = 1 + e^2$

so, $f(x) + f(2-x) = 1 + e^2$

$I = \int_0^2 f(x)\, dx$

$I = \int_0^2 f(2-x)\, dx$

$2I = \int_0^2 (f(x) + f(2-x))\, dx$

$2I = (1 + e^2)\int_0^2 dx$

$I = 1 + e^2$

26. Official Ans. by NTA (3)

Sol. $\int_{-a}^{0} (-2x+2)dx + \int_0^2 (x+2-x)dx + \int_2^a (2x-2)dx = 22$

$x^2 - 2x\Big|_0^{-a} + 2x\Big|_0^2 + x^2 - 2x\Big|_0^a = 22$

$a^2 + 2a + 4 + a^2 - 2a - (4 - 4) = 22$

$2a^2 = 18 \Rightarrow a = 3$

$\int_3^{-3} (x + [x])dx = -(-3 - 2 - 1 + 1 + 2) = 3$

27. Official Ans. by NTA (3)

Sol. $I = \int_{-1}^{1} x^2 e^{[x^3]} dx$

$= \int_{-1}^{0} x^2 e^{[x^3]} dx + \int_0^1 x^2 e^{[x^3]} dx$

$= \int_{-1}^{0} x^2 e^{-1} dx + \int_0^1 x^2 e^0 dx$

$= \frac{1}{e} \times \frac{x^3}{3}\Bigg|_{-1}^{0} + \frac{x^3}{3}\Bigg|_0^1$

$= \frac{1}{e} \times \left(0 - \left(\frac{-1}{3}\right)\right) + \frac{1}{3}$

$= \frac{1}{3e} + \frac{1}{3} = \frac{1+e}{3e}$

28. Official Ans. by NTA (1)

Sol. $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+...+\frac{n}{(2n-1)^2}\right]$

$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{n}{(n+r)^2}=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{n}{n^2+2nr+r^2}$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{(r/n)^2+2(r/n)+1}$

$=\int_0^1\frac{dx}{(x+1)^2}=\left[\frac{-1}{(x+1)}\right]_0^1=\frac{1}{2}$

29. Official Ans. by NTA (19)

Sol. $\int_{-2}^{2}3\left|x^2-x-2\right|dx$

$=3\int_{-2}^{2}\left|x^2-x-2\right|dx$

$=3\left[\int_{-2}^{-1}\left(x^2-x-2\right)dx+\int_{-1}^{2}-\left(x^2-x-2\right)dx\right]$

$=3\left[\left(\frac{x^3}{3}-\frac{x^2}{2}-2x\right)\Big|_{-2}^{-1}-\left(\frac{x^3}{3}-\frac{x^2}{2}-2x\right)\Big|_{-1}^{2}\right]$

$=3\left[7-\frac{2}{3}\right]$

$=19$

30. Official Ans. by NTA (3)

Sol. $f(x)=\int_1^x\frac{\log_e t}{(1+t)}dt$

$f\left(\frac{1}{x}\right)=\int_1^{1/x}\frac{\ell nt}{1+t}dt$, let $t=\frac{1}{y}$

$=+\int_1^x\frac{\ell ny}{1+y}.\frac{y}{y^2}dy$

$=\int_1^x\frac{\ell ny}{y(1+y)}dy$

hence

$f(x)+f\left(\frac{1}{x}\right)=\int_1^x\frac{(1+t)\ell nt}{t(1+t)}dt=\int_1^x\frac{\ell nt}{t}dt$

$=\frac{1}{2}\ell n^2(x)$

so $f(e)+f\left(\frac{1}{e}\right)=\frac{1}{2}$ …(3)

31. Official Ans. by NTA (1)

Sol. $f(x)=\int_0^x e^t f(t)dt+e^x \Rightarrow f(0)=1$

differentiating with respect to x

$f'(x) = e^x f(x) + e^x$

$f'(x) = e^x(f(x) + 1)$

$\int_0^x\frac{f'(x)}{f(x)+1}dx=\int_0^x e^x dx$

$\ell n\left(f(x)+1\right)\Big|_0^x=e^x\Big|_0^x$

$\ell n(f(x) + 1) - \ell n(f(0) + 1) = e^x - 1$

$\ell n\left(\frac{f(x)+1}{2}\right)=e^x-1$ {as $f(0) = 1$}

$f(x)=2e^{(e^x-1)}-1$

32. Official Ans. by NTA (1)

Sol. $I_{m,n}=\int_0^1 x^{m-1}(1-x)^{n-1}dx=I_{n,m}$

Now Let $x=\frac{1}{y+1}\Rightarrow dx=-\frac{1}{(y+1)^2}dy$

so

$I_{m,n}=-\int_\infty^0\frac{1}{(y+1)^{m-1}}\frac{y^{n-1}}{(y+1)^{n-1}}\frac{dy}{(y+1)^2}=\int_0^\infty\frac{y^{n-1}}{(1+y)^{m+n}}dy$

similarly $I_{m,n}=\int_0^\infty\frac{y^{m-1}}{(1+y)^{m+n}}dy$

Now $2I_{m,n}=\int_0^\infty\frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy$

$=\int_0^\infty\frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy$

$=\int_0^1\frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy+\underbrace{\int_1^\infty\frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy}_{\text{substitute } y=\frac{1}{t}}$

$\Rightarrow 2I_{m,n}=\int_0^1\frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy-\int_1^0\frac{t^{n-1}+t^{m-1}}{t^{m+n-2}}\frac{t^{m+n}}{(1+t)^{m+n}}\frac{dt}{t^2}$

$\Rightarrow$ Hence $2I_{m,n}=2\int_0^1\frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy\Rightarrow\alpha=1$

33. Official Ans. by NTA (2)

Sol. Put $2x = t \Rightarrow 2dx = dt$

$$\Rightarrow I = \frac{1}{2}\int_0^{2\pi}|\sin t|dt$$

$$= \int_0^{\pi}|\sin t|dt$$

$= 2$

34. Official Ans. by NTA (1)

Sol. $I = \int_{-\pi/2}^{\pi/2}\frac{\cos^2 x}{1+3^x}dx$ (using king)

$$I = \int_{-\pi/2}^{\pi/2}\frac{\cos^2 x}{1+3^{-x}}dx = \int_{-\pi/2}^{\pi/2}\frac{3^x\cos^2 x}{1+3^x}dx$$

$$2I = \int_{-\pi/2}^{\pi/2}\frac{(1+3^x)\cos^2 x}{1+3^x}dx$$

$$= \int_{-\pi/2}^{\pi/2}\cos^2 x dx = 2\int_0^{\pi/2}\cos^2 x\,dx$$

$$\Rightarrow I = \int_0^{\pi/2}\cos^2 x\,dx = \frac{\pi}{4}$$

35. Official Ans. by NTA (4)

Sol. $I_n = \int_{\pi/4}^{\pi/2}\cot^n x\,dx = \int_{\pi/4}^{\pi/2}\cot^{n-2}x(\operatorname{cosec}^2 x - 1)\,dx$

$$= -\left.\frac{\cot^{n-1}x}{n-1}\right]_{\pi/4}^{\pi/2} - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2}$$

$$\Rightarrow I_n + I_{n-2} = \frac{1}{n-1}$$

$$\Rightarrow I_2 + I_4 = \frac{1}{3}$$

$$I_3 + I_5 = \frac{1}{4}$$

$$I_4 + I_6 = \frac{1}{5}$$

$\therefore \frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}$ are in A.P.

36. Official Ans. by NTA (1)

Sol. $\sum_{n=1}^{100}\int_{n-1}^{n}e^{\{x\}}dx$, period of $\{x\} = 1$

$$\sum_{n=1}^{100}\int_0^1 e^{\{x\}}dx = \sum_{n=1}^{100}\int_0^1 e^x dx$$

$$\sum_{n=1}^{100}(e-1) = 100(e-1)$$

37. Official Ans by NTA (3)

Sol. $I = \int_0^{10}[x]\cdot e^{[x]-x+1}$

$$I = \int_0^1 0dx + \int_1^2 1\cdot e^{2-x} + \int_2^3 2\cdot e^{3-x} + ... + \int_9^{10} 9\cdot e^{10-x}dx$$

$$\Rightarrow I = \sum_{n=0}^{9}\int_n^{n+1} n\cdot e^{n+1-x}dx$$

$$= -\sum_{n=0}^{9} n\left(e^{n+1-x}\right)_n^{n+1}$$

$$= -\sum_{n=0}^{9} n\cdot(e^0 - e^1)$$

$$= (e-1)\sum_{n=0}^{9} n$$

$$= (e-1)\cdot\frac{9\cdot 10}{2}$$

$= 45(e-1)$

38. Official Ans by NTA (3)

Sol. $\int_0^1 (x^2 + bx + c)dx = 1$

$$\frac{1}{3} + \frac{b}{2} + c = 1 \quad \Rightarrow \quad \frac{b}{2} + c = \frac{2}{3}$$

$3b + 6c = 4$ ...(1)

$P(2) = 5$

$4 + 2b + c = 5$

$2b + c = 1$ ...(2)

From (1) & (2)

$b = \frac{2}{9}$ & $c = \frac{5}{9}$

$9(b + c) = 7$

39. Official Ans. by NTA (16)

Sol. $f(x) + f(x+1) = 2$

$\Rightarrow$ $f(x)$ is periodic with period = 2

$$I_1 = \int_0^8 f(x)dx = 4\int_0^2 f(x)dx$$

$$= 4\int_0^1 \left(f(x) + f(1+x)\right)dx = 8$$

Similarly $I_2 = 2 \times 2 = 4$

$I_1 + 2I_2 = 16$

40. Official Ans. by NTA (2)

Sol. $f(x) = e^{-x}\sin x$

Now, $F(x) = \int_0^x f(t)dt \quad \Rightarrow F'(x) = f(x)$

$$I = \int_0^1 \left(F'(x) + f(x)\right)e^x\,dx = \int_0^1 \left(f(x) + f(x)\right)\cdot e^x\,dx$$

$$= 2\int_0^1 f(x)\cdot e^x\,dx = 2\int_0^1 e^{-x}\sin x\cdot e^x\,dx$$

$$= 2\int_0^1 \sin x\,dx$$

$= 2(1 - \cos 1)$

$$I = 2\left\{1 - \left(1 - \frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 6} + \frac{1}{\lfloor 8}.........\right)\right\}$$

$$I = 1 - \frac{2}{\lfloor 4} + \frac{2}{\lfloor 6} - \frac{2}{\lfloor 9} + ...$$

$$1 - \frac{2}{\lfloor 4} < I < 1 - \frac{2}{\lfloor 4} + \frac{2}{\lfloor 6}$$

$$\frac{11}{12} < I < \frac{331}{360}$$

$$\Rightarrow I \in \left[\frac{11}{12}, \frac{331}{360}\right]$$

$$\Rightarrow I \in \left[\frac{330}{360}, \frac{331}{360}\right]$$ **Ans. (2)**

41. Official Ans. by NTA (1)

Sol. Let $I = \int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}}dx = \int_0^{10} \frac{[\sin 2\pi x]}{e^{\{x\}}}dx$

Function $f(x) = \frac{[\sin 2\pi x]}{e^{\{x\}}}$ is periodic with period '1'

Therefore

$$I = 10\int_0^1 \frac{[\sin 2\pi x]}{e^{\{x\}}}dx$$

$$= 10\int_0^1 \frac{[\sin 2\pi x]}{e^x}dx$$

$$= 10\left(\int_0^{1/2} \frac{[\sin 2\pi x]}{e^x}dx + \int_{1/2}^1 \frac{[\sin 2\pi x]}{e^x}dx\right)$$

$$= 10\left(0 + \int_{1/2}^1 \frac{(-1)}{e^x}dx\right)$$

$$= -10\int_{1/2}^1 e^{-x}\,dx$$

$$= 10\left(e^{-1} - e^{-1/2}\right)$$

Now,

$10\cdot e^{-1} - 10\cdot e^{-1/2} = \alpha e^{-1} + \beta e^{-1/2} + \gamma$ (given)

$\Rightarrow \alpha = 10, \beta = -10, \gamma = 0$

$\Rightarrow \alpha + \beta + \gamma = 0$ **Ans. (1)**

42. Official Ans. by NTA (1)

Sol. $I_n = \int_1^e x^{19}\left(\log|x|\right)^n dx$

$$I_n = \left|\left(\log|x|\right)^{19}\frac{x^{20}}{20}\right|_1^e - \int n(\log|x|)^{n-1}.\frac{1}{x}.\frac{x^{20}}{20}dx$$

$20I_n = e^{20} - nI_{n-1}$

$\therefore\ 20I_{10} = e^{20} - 10I_9$

$20I_9 = e^{20} - 9I_8$

$\Rightarrow 20I_{10} = 10I_9 + 9I_8$

$\alpha = 10, \beta = 9$

43. Official Ans. by NTA (Bonus)

Sol. $g(\alpha) = \int_{\frac{\pi}{6}}^{\pi/3} \frac{\sin^\alpha x}{(\sin^\alpha x + \cos^\alpha x)}$ .....(i)

$g(\alpha) = \int_{\frac{\pi}{6}}^{\pi/3} \frac{\cos^\alpha x}{(\sin^\alpha x + \cos^\alpha x)}$ .....(ii)

(1) + (2)

$2g(\alpha) = \frac{\pi}{6}$

$g(\alpha) = \frac{\pi}{12}$

Constant and even function

Due to typing mistake it must be bonus.

44. Official Ans. by NTA (512)

Sol. $I = 2\int_0^4 f(x^2)dx$ {Even funtion}

$= 2\int_0^4 \left(4x^3 - g(4-x)\right)dx$

$= 2\left(\frac{4x^4}{4}\Big|_0^4 - \int_0^4 g(4-x)dx\right)$

= 2(256 – 0) = 512

45. Official Ans. by NTA (3)

Sol. $\frac{1}{3} \le f(t) \le 1 \ \forall\ t \in [0,1]$

$0 \le f(t) \le \frac{1}{2} \ \forall\ t \in (1,3]$

Now, $g(3) = \int_0^3 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^3 f(t)\,dt$

$\because \int_0^1 \frac{1}{3}\,dt \le \int_0^1 f(t)\,dt \le \int_0^1 1.\,dt$ ....(1)

and $\int_1^3 0\,dt \le \int_1^3 f(1)\,dt \le \int_1^3 \frac{1}{2}\,dt$ ....(2)

Adding, we get

$\frac{1}{3} + 0 \le g(3) \le 1 + \frac{1}{2}(3-1)$

$\frac{1}{3} \le g(3) \le 2$