BINOMIAL THEOREM

1. The coefficient of x^{256} in the expansion of $(1-x)^{101}(x^2+x+1)^{100}$ is :

(1) ${}^{100}C_{16}$ (2) ${}^{100}C_{15}$

(3) $-{}^{100}C_{16}$ (4) $-{}^{100}C_{15}$

2. The number of rational terms in the binomial expansion of $\left(4^{\frac{1}{4}}+5^{\frac{1}{6}}\right)^{120}$ is _________.

3. For the natural numbers m, n, if $(1-y)^m(1+y)^n = 1 + a_1y + a_2y^2 + ... + a_{m+n}y^{m+n}$ and $a_1 = a_2 = 10$, then the value of $(m+n)$ is equal to :

(1) 88 (2) 64 (3) 100 (4) 80

4. For $k \in N$,

let $\dfrac{1}{\alpha(\alpha+1)(\alpha+2).......(\alpha+20)} = \sum_{K=0}^{20}\dfrac{A_k}{\alpha+k}$,

where $\alpha > 0$. Then the value of $100\left(\dfrac{A_{14}+A_{15}}{A_{13}}\right)^2$ is equal to ________.

5. If the constant term, in binomial expansion of $\left(2x^r+\dfrac{1}{x^2}\right)^{10}$ is 180, then r is equal to _______.

6. The number of elements in the set $\{n \in \{1,2,3,....,100\} \mid (11)^n > (10)^n + (9)^n\}$ is_______.

7. If b is very small as compared to the value of a, so that the cube and other higher powers of $\dfrac{b}{a}$ can be neglected in the identity $\dfrac{1}{a-b}+\dfrac{1}{a-2b}+\dfrac{1}{a-3b}+....+\dfrac{1}{a-nb} = \alpha n + \beta n^2 + \gamma n^3$, then the value of γ is :

(1) $\dfrac{a^2+b}{3a^3}$ (2) $\dfrac{a+b}{3a^2}$

(3) $\dfrac{b^2}{3a^3}$ (4) $\dfrac{a+b^2}{3a^3}$

8. The ratio of the coefficient of the middle term in the expansion of $(1+x)^{20}$ and the sum of the coefficients of two middle terms in expansion of $(1+x)^{19}$ is ________.

9. The term independent of 'x' in the expansion of $\left(\dfrac{x+1}{x^{2/3}-x^{1/3}+1}-\dfrac{x-1}{x-x^{1/2}}\right)^{10}$, where $x \neq 0, 1$ is equal to_______.

10. The sum of all those terms which are rational numbers in the expansion of $\left(2^{1/3}+3^{1/4}\right)^{12}$ is:

(1) 89 (2) 27 (3) 35 (4) 43

11. If the greatest value of the term independent of 'x' in the expansion of $\left(x\sin\alpha + a\dfrac{\cos\alpha}{x}\right)^{10}$ is $\dfrac{10!}{(5!)^2}$, then the value of 'a' is equal to :

(1) –1 (2) 1 (3) –2 (4) 2

12. The lowest integer which is greater than $\left(1+\dfrac{1}{10^{100}}\right)^{10^{100}}$ is ________.

(1) 3 (2) 4 (3) 2 (4) 1

13. Let $n \in \mathbf{N}$ and [x] denote the greatest integer less than or equal to x. If the sum of (n + 1) terms ${}^nC_0, 3\cdot{}^nC_1, 5\cdot{}^nC_2, 7\cdot{}^nC_3, ...$ is equal to $2^{100}\cdot 101$, then $2\left[\dfrac{n-1}{2}\right]$ is equal to _________.

14. If the co-efficient of x^7 and x^8 in the expansion of $\left(2+\dfrac{x}{3}\right)^n$ are equal, then the value of n is equal to _______.

15. If the coefficients of x^7 in $\left(x^2+\frac{1}{bx}\right)^{11}$ and x^{-7} in $\left(x-\frac{1}{bx^2}\right)^{11}$, $b \neq 0$, are equal, then the value of b is equal to:

(1) 2 (2) –1 (3) 1 (4) –2

16. A possible value of 'x', for which the ninth term in the expansion of $\left\{3^{\log_3\sqrt{25^{x-1}+7}}+3^{\left(-\frac{1}{8}\right)\log_3\left(5^{x-1}+1\right)}\right\}^{10}$ in the increasing powers of $3^{\left(-\frac{1}{8}\right)\log_3\left(5^{x-1}+1\right)}$ is equal to 180, is :

(1) 0 (2) –1 (3) 2 (4) 1

17. If ${}^{20}C_r$ is the co-efficient of x^r in the expansion of $(1 + x)^{20}$, then the value of $\sum_{r=0}^{20} r^2\,{}^{20}C_r$ is equal to :

(1) 420×2^{19} (2) 380×2^{19}
(3) 380×2^{18} (4) 420×2^{18}

18. If $1\cdot{}^1P_1 + 2\cdot{}^2P_2 + 3\cdot{}^3P_3 + \ldots + 15\cdot{}^{15}P_{15} = {}^qP_r - s$, $0 \le s \le 1$, then ${}^{q+s}C_{r-s}$ is equal to ________.

19. Let $\binom{n}{k}$ denotes ${}^nC_k$ and $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \binom{n}{k}, & \text{if } 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$

If $A_k = \sum_{i=0}^{9}\binom{9}{i}\begin{bmatrix} 12 \\ 12-k+i \end{bmatrix} + \sum_{i=0}^{8}\binom{8}{i}\begin{bmatrix} 13 \\ 13-k+i \end{bmatrix}$ and $A_4 - A_3 = 190\,p$, then p is equal to :

20. If $0 < x < 1$, then $\frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{4}x^4 + \ldots$, is equal to :

(1) $x\left(\frac{1+x}{1-x}\right) + \log_e(1-x)$

(2) $x\left(\frac{1-x}{1+x}\right) + \log_e(1-x)$

(3) $\frac{1-x}{1+x} + \log_e(1-x)$

(4) $\frac{1+x}{1-x} + \log_e(1-x)$

21. $\sum_{k=0}^{20}\left({}^{20}C_k\right)^2$ is equal to :

(1) ${}^{40}C_{21}$ (2) ${}^{40}C_{19}$ (3) ${}^{40}C_{20}$ (4) ${}^{41}C_{20}$

22. $3 \times 7^{22} + 2 \times 10^{22} - 44$ when divided by 18 leaves the remainder __________.

23. If $\left(\frac{3^6}{4^4}\right)k$ is the term, independent of x, in the binomial expansion of $\left(\frac{x}{4}-\frac{12}{x^2}\right)^{12}$, then k is equal to _________.

24. If the coefficient of a^7b^8 in the expansion of $(a + 2b + 4ab)^{10}$ is $K.2^{16}$, then K is equal to _____.

25. If the sum of the coefficients in the expansion of $(x + y)^n$ is 4096, then the greatest coefficient in the expansion is _______.

26. If $n \ge 2$ is a positive integer, then the sum of the series ${}^{n+1}C_2 + 2({}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots + {}^nC_2)$ is:

(1) $\frac{n(n-1)(2n+1)}{6}$ (2) $\frac{n(n+1)(2n+1)}{6}$

(3) $\frac{n(2n+1)(3n+1)}{6}$ (4) $\frac{n(n+1)^2(n+2)}{12}$

27. For integers n and r, let $\binom{n}{r} = \begin{cases} {}^nC_r, & \text{if } n \ge r \ge 0 \\ 0, & \text{otherwise} \end{cases}$

The maximum value of k for which the sum $\sum_{i=0}^{k}\binom{10}{i}\binom{15}{k-i} + \sum_{i=0}^{k+1}\binom{12}{i}\binom{13}{k+1-i}$ exists, is equal to ________.

28. The value of $-{}^{15}C_1 + 2.{}^{15}C_2 - 3.{}^{15}C_3 + \ldots\ldots$ $-15.{}^{15}C_{15} + {}^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 + \ldots + {}^{14}C_{11}$ is :

(1) $2^{16} - 1$ (2) $2^{13} - 14$
(3) 2^{14} (4) $2^{13} - 13$

29. If the remainder when x is divided by 4 is 3, then the remainder when $(2020 + x)^{2022}$ is divided by 8 is _______.

30. Let $m,n \in N$ and gcd $(2,n) = 1$. If

$$30\binom{30}{0}+29\binom{30}{1}+...+2\binom{30}{28}+1\binom{30}{29}=n.2^m,$$

then $n + m$ is equal to

(Here $\binom{n}{k}={}^nC_k$)

31. The maximum value of the term independent of 't' in the expansion of $\left(tx^{\frac{1}{5}}+\frac{(1-x)^{\frac{1}{10}}}{t}\right)^{10}$ where $x \in (0,1)$ is :

(1) $\frac{10!}{\sqrt{3}(5!)^2}$ (2) $\frac{2.10!}{3\sqrt{3}(5!)^2}$

(3) $\frac{2.10!}{3(5!)^2}$ (4) $\frac{10!}{3(5!)^2}$

32. Let n be a positive integer. Let

$$A=\sum_{k=0}^{n}(-1)^k\,{}^nC_k\left[\left(\frac{1}{2}\right)^k+\left(\frac{3}{4}\right)^k+\left(\frac{7}{8}\right)^k+\left(\frac{15}{16}\right)^k+\left(\frac{31}{32}\right)^k\right]$$

If $63A = 1-\frac{1}{2^{30}}$, then n is equal to _______.

33. If n is the number of irrational terms in the expansion of $(3^{1/4} + 5^{1/8})^{60}$, then$(n - 1)$ is divisible by :

(1) 26 (2) 30 (3) 8 (4) 7

34. Let [x] denote greatest integer less than or equal to x. If for $n \in \mathbb{N}$, $(1-x+x^3)^n=\sum_{j=0}^{3n}a_j x^j$, then

$$\sum_{j=0}^{\left[\frac{3n}{2}\right]}a_{2j}+4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]}a_{2j}+1$$ is equal to :

(1) 2 (2) 2^{n-1} (3) 1 (4) n

35. The value of $\sum_{r=0}^{6}\left({}^6C_r\cdot{}^6C_{6-r}\right)$ is equal to :

(1) 1124 (2) 1324 (3) 1024 (4) 924

36. Let the coefficients of third, fourth and fifth terms in the expansion of $\left(x+\frac{a}{x^2}\right)^n$, $x \neq 0$, be in the ratio 12 : 8 : 3. Then the term independent of x in the expansion, is equal to _______.

37. Two dices are rolled. If both dices have six faces numbered 1,2,3,5,7 and 11, then the probability that the sum of the numbers on the top faces is less than or equal to 8 is :

(1) $\frac{4}{9}$ (2) $\frac{17}{36}$ (3) $\frac{5}{12}$ (4) $\frac{1}{2}$

38. If the fourth term in the expansion of $(x+x^{\log_2 x})^7$ is 4480, then the value of x where $x \in N$ is equal to :

(1) 2 (2) 4 (3) 3 (4) 1

39. The value of $4+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{4+.......\infty}}}}$ is :

(1) $2+\frac{2}{5}\sqrt{30}$ (2) $2+\frac{4}{\sqrt{5}}\sqrt{30}$

(3) $4+\frac{4}{\sqrt{5}}\sqrt{30}$ (4) $5+\frac{2}{5}\sqrt{30}$

40. If $(2021)^{3762}$ is divided by 17, then the remainder is _______ .

41. Let $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 +...+ a_{40}x^{40}$. then $a_1 + a_3 + a_5 +... + a_{37}$ is equal to

(1) $2^{20}(2^{20} - 21)$ (2) $2^{19}(2^{20} - 21)$

(3) $2^{19}(2^{20} + 21)$ (4) $2^{20}(2^{20} + 21)$

42. If $\sum_{r=1}^{10}r!\left(r^3+6r^2+2r+5\right)=\alpha(11!)$, then the value of α is equal to _______.

43. Let P(x) be a real polynomial of degree 3 which vanishes at x = –3. Let P(x) have local minima at x = 1, local maxima at x = –1 and $\int_{-1}^{1} P(x)dx = 18$, then the sum of all the coefficients of the polynomial P(x) is equal to _______.

44. Let ^nC_r denote the binomial coefficient of x^r in the expansion of $(1 + x)^n$.

If $\sum_{k=0}^{10}(2^2+3k)\,{}^nC_k = \alpha.3^{10}+\beta.2^{10}$, α, β ∈ R, then α + β is equal to ______.

SOLUTION

1. Official Ans. by NTA (2)

Sol. $(1-x)^{100} \cdot (x^2+x+1)^{100} \cdot (1-x)$

$=\left((1-x)(x^2+x+1)\right)^{100}(1-x)$

$=(1^3-x^3)^{100}(1-x)$

$=(1-x^3)^{100}(1-x)$

$=\underbrace{(1-x^3)^{100}}_{\text{No term of } x^{256}} - \underbrace{x(1-x^3)^{100}}_{\text{We find cofficient of } x^{255}}$

Required coefficient $(-1)\times(-{}^{100}C_{85})$

$= {}^{100}C_{85} = {}^{100}C_{15}$

2. Official Ans. by NTA (21)

Sol. $\left(4^{1/4}+5^{1/6}\right)^{120}$

$T_{r+1} = {}^{120}C_r (2^{1/2})^{120-r}(5)^{r/6}$

for rational terms $r = 6\lambda$ $\quad 0 \le r \le 120$

so total no of forms are 21.

3. Official Ans. by NTA (4)

Sol. $(1-y)^m (1+y)^n$

Coefficient of y $(a_1) = 1.{}^nC_1 + {}^mC_1(-1)$

$= n - m = 10$ .....(1)

Coefficient of y^2 (a_2)

$= 1.\,{}^nC_2 - {}^mC_1.\,{}^nC_1. + 1.{}^mC_2 = 10$

$=\frac{n(n-1)}{2} - m.n + \frac{m(m-1)}{2} = 10$

$m^2 + n^2 - 2mn - (n+m) = 20$

$(n-m)^2 - (n+m) = 20$

$n + m = 80$ ..... (2)

By equation (1) & (2)

$m = 35, n = 45$

4. Official Ans. by NTA (9)

Sol. $\frac{1}{\alpha(\alpha+1).....(\alpha+20)} = \sum_{k=0}^{20}\frac{A_k}{\alpha+k}$

$A_{14} = \frac{1}{(-14)(-13).....(-1)(1).....(6)} = \frac{1}{14!\cdot 6!}$

$A_{15} = \frac{1}{(-15)(-14).....(-1)(1).....(5)} = \frac{1}{15!\cdot 5!}$

$A_{13} = \frac{1}{(-13).....(-1)(1).....(7)} = \frac{-1}{13!\cdot 7!}$

$\frac{A_{14}}{A_{13}} = \frac{1}{14!\cdot 6!}\times -13!\times 7! = \frac{-7}{14} = -\frac{1}{2}$

$\frac{A_{15}}{A_{13}} = -\frac{1}{15!\times 5!}\times -13!\times 7! = \frac{42}{15\times 14} = \frac{1}{5}$

$100\left(\frac{A_{14}}{A_{13}}+\frac{A_{15}}{A_{13}}\right)^2 = 100\left(-\frac{1}{2}+\frac{1}{5}\right)^2 = 9$

5. Official Ans. by NTA (8)

Sol. $\left(2x^r + \frac{1}{x^2}\right)^{10}$

General term = ${}^{10}C_R (2x^2)^{10-R} x^{-2R}$

$\Rightarrow 2^{10-R}\,{}^{10}C_R = 180$ .... (1)

& $(10-R)\,r - 2R = 0$

$r = \frac{2R}{10-R}$

$r = \frac{2(R-10)}{10-R} + \frac{20}{10-R}$

$\Rightarrow r = -2 + \frac{20}{10-R}$ .... (2)

R = 8 or 5 reject equation (1) not satisfied

At R = 8

$2^{10-R}\,{}^{10}C_R = 180 \Rightarrow \boxed{r=8}$

6. Official Ans. by NTA (96)

Sol. $11^n > 10^n + 9^n$

$\Rightarrow 11^n - 9^n > 10^n$

$\Rightarrow (10 + 1)^n - (10 - 1)^n > 10^n$

$\Rightarrow \{ {}^nC_1 \cdot 10^{n-1} + {}^nC_3 10^{n-0} + {}^nC_5 10^{n-5} + \cdots\} > 10^n$

$\Rightarrow 2n \cdot 10^{n-1} + 2\{ {}^nC_3 10^{n-3} + {}^nC_5 10^{n-5} + \cdots\} > 10^n$ .... (1)

For n = 5

$10^5 + 2\{ {}^5C_3 10^2 + {}^5C_5\} > 10^5$ (True)

For n = 6, 7, 8, ….. 100

$2n10^{n-1} > 10^n$

$\Rightarrow 2n10^{n-1} + 2\ \{ {}^nC_3 10^{n-3} + {}^nC_5 10^{n-5} + \cdots\} > 10^n$

$\Rightarrow 11^n - 9^n > 10^n$ For n = 5, 6, 7, ….. 100

For n = 4, Inequality (1) is not satisfied

$\Rightarrow$ Inequality does not hold good for

N = 1, 2, 3, 4

So, required number of elements

= 96

7. Official Ans. by NTA (3)

Sol. $(a - b)^{-1} + (a - 2b)^{-1} + \ldots. + (a - nb)^{-1}$

$$= \frac{1}{a}\sum_{r=1}^{n}\left(1 - \frac{rb}{a}\right)^{-1}$$

$$= \frac{1}{a}\sum_{r=1}^{n}\left\{\left(1 + \frac{rb}{a} + \frac{r^2b^2}{a^2}\right) + (\text{terms to be neglected})\right\}$$

$$= \frac{1}{a}\left[n + \frac{n(n+1)}{2}.\frac{b}{a} + \frac{n(n+1)(2n+1)}{6}.\frac{b^2}{a^2}\right]$$

$$= \frac{1}{a}\left[n^3\left(\frac{b^2}{3a^2}\right) + \ldots.\right]$$

So $\gamma = \dfrac{b^2}{3a^3}$

8. Official Ans. by NTA (1)

Sol. Coeff. of middle term in $(1 + x)^{20} = {}^{20}C_{10}$

& Sum of Coeff. of two middle terms in

$(1 + x)^{19} = {}^{19}C_9 + {}^{19}C_{10}$

So required ratio $= \dfrac{{}^{20}C_{10}}{{}^{19}C_9 + {}^{19}C_{10}} = \dfrac{{}^{20}C_{10}}{{}^{20}C_{10}} = 1$

9. Official Ans. by NTA (210)

Sol. $$\left(\left(x^{1/3} + 1\right) - \left(\frac{x^{1/2} + 1}{x^{1/2}}\right)\right)^{10}$$

$$= \left(x^{1/3} - \frac{1}{x^{1/2}}\right)^{10}$$

Now General Term

$$T_{r+1} = {}^{10}C_r\left(x^{1/3}\right)^{10-r}.\left(-\frac{1}{x^{1/2}}\right)^r$$

For independent term

$\dfrac{10 - r}{3} - \dfrac{r}{2} = 0 \Rightarrow r = 4$

$\Rightarrow T_5 = {}^{10}C_4 = 210$

10. Official Ans. by NTA (4)

Sol. $T_{r+1} = {}^{12}C_r\left(2^{1/3}\right)^r.\left(3^{1/4}\right)^{12-r}$

T_{r+1} will be rational number

when r = 0, 3, 6, 9, 12

& r = 0, 4, 8, 12

$\Rightarrow$ r = 0, 12

$T_1 + T_{13} = 1 \times 3^3 + 1 \times 2^4 \times 1$

= 24 + 16 = 43

11. Official Ans. by NTA (4)

Sol. $$T_{r+1} = {}^{10}C_r(x\sin\alpha)^{10-r}\left(\frac{a\cos\alpha}{x}\right)^r$$

r = 0, 1, 2, …, 10

T_{r+1} will be independent of x

when 10 – 2r = 0 $\Rightarrow$ r = 5

$$T_6 = {}^{10}C_5(x\sin\alpha)^5 \times \left(\frac{a\cos\alpha}{x}\right)^5$$

$$= {}^{10}C_5 \times a^5 \times \frac{1}{2^5}(\sin 2\alpha)^5$$

will be greatest when $\sin 2\alpha = 1$

$\Rightarrow\ {}^{10}C_5\dfrac{a^5}{2^5} = {}^{10}C_5 \Rightarrow a = 2$

12. Official Ans. by NTA (1)

Sol. Let $P=\left(1+\frac{1}{10^{100}}\right)^{10^{100}}$,

Let $x = 10^{100}$

$\Rightarrow P=\left(1+\frac{1}{x}\right)^{x}$

$\Rightarrow P=1+(x)\left(\frac{1}{x}\right)+\frac{(x)(x-1)}{\lfloor 2}\cdot\frac{1}{x^2}$

$+\frac{(x)(x-1)(x-2)}{\lfloor 3}\cdot\frac{1}{x^3}+\ldots$

(upto $10^{100}+1$ terms)

$\Rightarrow P=1+1+\left(\frac{1}{\lfloor 2}-\frac{1}{\lfloor 2\, x^2}\right)+\left(\frac{1}{\lfloor 3}-\ldots\right)+\ldots$ so on

$\Rightarrow P=2+\left(\text{Positive value less then } \frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots\right)$

Also $e=1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots$

$\Rightarrow \frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots=e-2$

$\Rightarrow$ P = 2 + (positive value less then e – 2)

$\Rightarrow P \in (2, 3)$

$\Rightarrow$ least integer value of P is 3

13. Official Ans. by NTA (98)

Sol. $1.{}^nC_0 + 3.{}^nC_1 + 5.{}^nC_2 + \ldots + (2n+1).{}^nC_n$

$T_r = (2r+1){}^nC_r$

$S = \Sigma T_r$

$S = \Sigma(2r+1){}^nC_r = \Sigma 2r\,{}^nC_r + \Sigma\,{}^nC_r$

$S = 2(n.2^{n-1}) + 2^n = 2^n(n+1)$

$2^n(n+1) = 2^{100}.101 \Rightarrow n = 100$

$2\left[\frac{n-1}{2}\right]=2\left[\frac{99}{2}\right]=98$

14. Official Ans. by NTA (55)

Sol. ${}^nC_7 2^{n-7}\frac{1}{3^7}={}^nC_8 2^{n-8}\frac{1}{3^8}$

$\Rightarrow n-7=48 \Rightarrow n=55$

15. Official Ans. by NTA (3)

Sol. Coefficient of x^7 in $\left(x^2+\frac{1}{bx}\right)^{11}$

${}^{11}C_r\,(x^2)^{11-r}.\left(\frac{1}{bx}\right)^r$

${}^{11}C_r x^{22-3r}.\frac{1}{b^r}$

$22-3r=7$

$r=5$

$\therefore\ {}^{11}C_5.\frac{1}{b^5}.x^7$

Coefficient of x^{-7} in $\left(x-\frac{b}{bx^2}\right)^{11}$

${}^{11}C_r\,(x)^{11-r}.\left(-\frac{1}{bx^2}\right)^r$

${}^{11}C_r\,x^{11-3r}.\frac{(-1)^r}{b^r}$

$11-3r=-7\ \therefore r=6$

${}^{11}C_6.\frac{1}{b^6}x^{-7}$

${}^{11}C_5.\frac{1}{b^5}={}^{11}C_6.\frac{1}{b^6}$

Since $b\neq 0\ \therefore b=1$

16. Official Ans. by NTA (4)

Sol. ${}^{10}C_8\left(25^{(x-1)}+7\right)\times\left(5^{(x-1)}+1\right)^{-1}=180$

$\Rightarrow \frac{25^{x-1}+7}{5^{(x-1)}+1}=4$

$\Rightarrow \frac{t^2+7}{t+1}=4;$

$\Rightarrow t=1,\ 3=5^{x-1}$

$\Rightarrow x-1=0$ (one of the possible value).

$\Rightarrow x=1$

17. Official Ans. by NTA (4)

Sol. $\sum_{r=0}^{20} r^2.{}^{20}C_r$

$\sum(4(r-1)+r).{}^{20}C_r$

$\sum r(r-1).\frac{20\times19}{r(r-1)}.{}^{18}C_r + r.\frac{20}{r}.\sum {}^{19}C_{r-1}$

$\Rightarrow 20 \times 19.2^{18} + 20.2^{19}$

$\Rightarrow 420 \times 2^{18}$

18. Official Ans. by NTA (136)

Sol. ${}^1P_1 + 2\cdot{}^2P_2 + 3\cdot{}^3P_3 + \ldots + 15 \cdot {}^{15}P_{15}$

$= 1! + 2.2! + 3.3! + \ldots.15 \times 15!$

$= \sum_{r=1}^{15}(r+1-1)r!$

$= \sum_{r=1}^{15}(r+1)! - (r)!$

$= 16! - 1$

$= {}^{16}P_{16} - 1$

$\Rightarrow q = r = 16, s = 1$

${}^{q+s}C_{r-s} = {}^{17}C_{15} = 136$

19. Official Ans. by NTA (49)

Sol. $A_k = \sum_{i=0}^{9} {}^9C_i \; {}^{12}C_{k-i} + \sum_{i=0}^{8} {}^8C_i \; {}^{13}C_{k-i}$

$A_k = {}^{21}C_k + {}^{21}C_k = 2 \cdot {}^{21}C_k$

$A_4 - A_3 = 2\left({}^{21}C_4 - {}^{21}C_3\right) = 2(5985 - 1330)$

$190\,p = 2(5985 - 1330) \Rightarrow p = 49.$

20. Official Ans. by NTA (1)

Sol. Let $t = \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{4}x^4 + \ldots.\infty$

$= \left(2-\frac{1}{2}\right)x^2 + \left(2-\frac{1}{3}\right)x^3 + \left(2-\frac{1}{4}\right)x^4 + \ldots.\infty$

$= 2(x^2 + x^3 + x^4 + \ldots\infty) - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\infty\right)$

$= \frac{2x^2}{1-x} - (\ell n(1-x) - x)$

$\Rightarrow t = \frac{2x^2}{1-x} + x - \ell n(1-x)$

$\Rightarrow t = \frac{x(1+x)}{1-x} - \ell n(1-x)$

21. Official Ans. by NTA (3)

Sol. $\sum_{k=0}^{20} {}^{20}C_k . {}^{20}C_{20-k}$

sum of suffix is const. so summation will be ${}^{40}C_{20}$

22. Official Ans. by NTA (15)

Sol. $3(1+6)^{22} + 2\cdot(1+9)^{22} - 44 = (3 + 2 - 44) = 18\,.I$

$= -39 + 18.I$

$= (54 - 39) + 18(I - 3)$

$= 15 + 18\,I_1$

$\Rightarrow$ Remainder = 15.

23. Official Ans. by NTA (55)

Sol. $\left(\frac{x}{4} - \frac{12}{x^2}\right)^{12}$

$T_{r+1} = (-1)^r \,.\, {}^{12}C_r \left(\frac{x}{4}\right)^{12-r}\left(\frac{12}{x^2}\right)^r$

$T_{r+1} = (-1)^r \,.\, {}^{12}C_r \left(\frac{1}{4}\right)^{12-r}(12)^r.(x)^{12-3r}$

Term independent of x $\Rightarrow 12 - 3r = 0 \Rightarrow r = 4$

$T_5 = (-1)^4 \,.\, {}^{12}C_4 \left(\frac{1}{4}\right)^8 (12)^4 = \frac{3^6}{4^4}.k$

$\Rightarrow k = 55$

24. Official Ans. by NTA (315)

Sol. $\frac{10!}{\alpha!\beta!\gamma!}a^{\alpha}(2b)^{\beta}.(4ab)^{\gamma}$

$\frac{10!}{\alpha!\beta!\gamma!}a^{\alpha+\gamma}.b^{\beta+\gamma}.2^{\beta}.4^{\gamma}$

$\alpha+\beta+\gamma=10$ …(1)

$\alpha+\gamma=7$ …(2)

$\beta+\gamma=8$ …(3)

$(2)+(3)-(1)\Rightarrow\gamma=5$

$\alpha=2$

$\beta=3$

so coefficients $=\frac{10!}{2!3!5!}2^3.2^{10}$

$=\frac{10\times9\times8\times7\times6\times5}{2\times3\times2\times5!}\times2^{13}$

$=315\times2^{16}\Rightarrow k=315$

25. Official Ans. by NTA (924)

Sol. $(x+y)^n\Rightarrow2^n=4096$ $\quad 2^{10}=1024\times2$

$\Rightarrow2^n=2^{12}$ $\quad 2^{11}=2048$

$n=12$ $\quad 2^{12}=\underline{4096}$

$^{12}C_6=\frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2\times1}$

$=11\times3\times4\times7$

$=924$

26. Official Ans. by NTA (2)

Sol. $^{n+1}C_2+2\left(^2C_2+{}^3C_2+{}^4C_2+........+{}^nC_2\right)$

$^{n+1}C_2+2\left(^3C_3+{}^3C_2+{}^4C_2+........+{}^nC_2\right)$

$\left\{\text{use } {}^nC_{r+1}+{}^nC_r={}^{n+1}C_r\right\}$

$={}^{n+1}C_2+2\left(^4C_3+{}^4C_2+{}^5C_3+.......+{}^nC_2\right)$

$={}^{n+1}C_2+2\left(^5C_3+{}^5C_2+.......+{}^nC_2\right)$

⋮ ⋮ ⋮ ⋮

$={}^{n+1}C_2+2\left(^nC_3+{}^nC_2\right)$

$={}^{n+1}C_2+2.{}^{n+1}C_3$

$=\frac{(n+1)n}{2}+2.\frac{(n+1)(n)(n-1)}{2.3}$

$=\frac{n(n+1)(2n+1)}{6}$

27. Official Ans. by NTA (BONUS)

Sol. **Bonus**

$\sum_{i=0}^{k}\binom{10}{i}\binom{15}{k-i}+\sum_{i=0}^{k+1}\binom{12}{i}\binom{13}{k+1-i}$

$^{25}C_k+{}^{25}C_{k+1}$

$^{26}C_{k+1}$

as ^nC_r is defined for all values of n as will as r

so $^{26}C_{k+1}$ always exists

Now k is unbounded so maximum value is not defined.

28. Official Ans. by NTA (2)

Sol. $(-{}^{15}C_1+2.{}^{15}C_2-3.{}^{15}C_3+......-15.{}^{15}C_{15})$
$+({}^{14}C_1+{}^{14}C_3+...+{}^{14}C_{11})$

$=\sum_{r=1}^{15}(-1)^r.r\,{}^{15}C_r+\left({}^{14}C_1+{}^{14}C_3+...+{}^{14}C_{11}+{}^{14}C_{13}\right)-{}^{14}C_3$

$=\sum_{r=1}^{15}(-1)^r15.{}^{14}C_{r-1}+2^{13}-14$

$=15(-{}^{14}C_0+{}^{14}C_1........-{}^{14}C_{14})+2^{13}-14$

29. Official Ans. by NTA (1)

Sol. $x = 4k + 3$

$\therefore (2020 + x)^{2022} = (2020 + 4k + 3)^{2022}$

$= (4(505 + k) + 3)^{2022}$

$= (4\lambda + 3)^{2022} = (16\lambda^2 + 24\lambda + 9)^{1011}$

$= (8(2\lambda^2 + 3\lambda + 1) + 1)^{1011}$

$= (8p + 1)^{1011}$

$\therefore$ Remainder when divided by 8 = 1

$= 2^{13} - 14$

30. Official Ans. by NTA (45)

Sol. $30(^{30}C_0) + 29(^{30}C_1) + ... + 2(^{30}C_{28}) + 1(^{30}C_{29})$

$= 30(^{30}C_{30}) + 29(^{30}C_{29}) + ... + 2(^{30}C_2) + 1(^{30}C_1)$

$$= \sum_{r=1}^{30} r\left(^{30}C_r\right)$$

$$= \sum_{r=1}^{30} r\left(\frac{30}{r}\right)\left(^{29}C_{r-1}\right)$$

$$= 30\sum_{r=1}^{30} {}^{29}C_{r-1}$$

$$= 30\left(^{29}C_0 + {}^{29}C_1 + {}^{29}C_2 + ... + {}^{29}C_{29}\right)$$

$$= 30\left(2^{29}\right) = 15(2)^{30} = n(2)^m$$

$\therefore n = 15, m = 30$

31. Official Ans. by NTA (2)

Sol. Term independent of t will be the middle term due to exect same magnitude but opposite sign powers of t in the binomial expression given

$$\text{so } T_6 = {}^{10}C_5\left(tx^2 5\right)^5\left(\frac{(1-x)^{\frac{1}{10}}}{t}\right)^5$$

$T_6 = f(x) = {}^{10}C_5\left(x\sqrt{1-x}\right)$; for maximum

$$f'(x) = 0 \Rightarrow x = \frac{2}{3} \;\&\; f''\left(\frac{2}{3}\right) < 0$$

$$\text{so } f(x)_{max.} = {}^{10}C_5\left(\frac{2}{3}\right).\frac{1}{\sqrt{3}}$$

32. Official Ans by NTA (6)

Sol. $$A = \sum_{k=0}^{n} {}^nC_k\left[\left(-\frac{1}{2}\right)^k + \left(\frac{-3}{4}\right)^k + \left(\frac{-7}{8}\right)^k + \left(\frac{-15}{16}\right)^k + \left(\frac{-37}{32}\right)^k\right]$$

$$A = \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \left(1-\frac{15}{16}\right)^n + \left(1-\frac{31}{32}\right)^n$$

$$A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{16^n} + \frac{1}{32^n}$$

$$A = \frac{1}{2^n}\left(\frac{1-\left(\frac{1}{2^n}\right)^5}{1-\frac{1}{2^n}}\right) \Rightarrow A = \frac{\left(1-\frac{1}{2^{5n}}\right)}{(2^n-1)}$$

$(2^n - 1)A = 1 - \dfrac{1}{2^{5n}}$, Given $63A = 1 - \dfrac{1}{2^{30}}$

Clearly $5n = 30$

$n = 6$

33. Official Ans. by NTA (1)

Sol. $(3^{1/4} + 5^{1/8})^{60}$

$^{60}C_r(3^{1/4})^{60-r}.(5^{1/8})^r$

$^{60}C_r(3)^{\frac{60-r}{4}}.5^{\frac{r}{8}}$

For rational terms.

$\frac{r}{8} = k;\quad 0 \le r \le 60$

$0 \le 8k \le 60$

$0 \le k \le \frac{60}{8}$

$0 \le k \le 7.5$

$k = 0, 1, 2, 3, 4, 5, 6, 7$

$\frac{60-8k}{4}$ is always divisible by 4 for all value of k.

Total rational terms = 8

Total terms = 61

irrational terms = 53

$n - 1 = 53 - 1 = 52$

52 is divisible by 26.

34. Official Ans. by NTA (3)

Sol. $(1-x+x^3)^n = \sum_{j=0}^{3n} a_j x^j$

$(1-x+x^3)^n = a_0 + a_1x + a_2x^2 \ldots\ldots + a_{3n}x^{3n}$

$\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j}$ = Sum of $a_0 + a_2 + a_4 \ldots\ldots$

$\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1}$ = Sum of $a_1 + a_3 + a_5 \ldots\ldots$

put x = 1

$1 = a_0 + a_1 + a_2 + a_3 \ldots\ldots + a_{3n}$ ....(A)

Put x = –1

$1 = a_0 - a_1 + a_2 - a_3 \ldots\ldots + (-1)^{3n}a_{3n}$ ....(B)

Solving (A) and (B)

$a_0 + a_2 + a_4 \ldots = 1$

$a_1 + a_3 + a_5 \ldots = 0$

$\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} + 4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1} = 1$

35. Official Ans. by NTA (4)

Sol. $\sum_{r=0}^{6} {}^6C_r \cdot {}^6C_{6-r}$

$= {}^6C_0 \cdot {}^6C_6 + {}^6C_1 \cdot {}^6C_5 + \ldots\ldots + {}^6C_6 \cdot {}^6C_0$

Now,

$(1+x)^6(1+x)^6$

$= \left({}^6C_0 + {}^6C_1x + {}^6C_2x^2 + \ldots\ldots + {}^6C_6x^6\right)$

$\left({}^6C_0 + {}^6C_1x + {}^6C_2x^2 + \ldots\ldots + {}^6C_6x^6\right)$

Comparing coefficeint of x^6 both sides

${}^6C_0 \cdot {}^6C_6 + {}^6C_1 + {}^6C_5 + \ldots\ldots + {}^6C_6 \cdot {}^6C_0 = {}^{12}C_6$

$= 924$

Ans.(4)

36. Official Ans. by NTA (4)

Sol. $T_{r+1} = {}^nC_r(x)^{n-r}\left(\frac{a}{x^2}\right)^r$

$= {}^nC_r\, a^r x^{n-3r}$

${}^nC_2\, a^2 : {}^nC_3\, a^3 : {}^nC_4\, a^4 = 12 : 8 : 3$

After solving

$n = 6,\ a = \frac{1}{2}$

For term independent of 'x' $\Rightarrow n = 3r$

$r = 2$

$\therefore$ Coefficient is ${}^6C_2\left(\frac{1}{2}\right)^2 = \frac{15}{4}$

Nearest integer is 4.

37. Official Ans. by NTA (2)

Sol. n(E) = 5 + 4 + 4 + 3 + 1 = 17

So, P (E) = $\frac{17}{36}$

38. Official Ans. by NTA (1)

Sol. ${}^7C_3x^4\ x^{(3\log_2^x)} = 4480$

$\Rightarrow x^{(4+3\log_2^x)} = 2^7$

$\Rightarrow (4+3t)t = 7; t = \log_2^x$

$\Rightarrow t = 1, \frac{-7}{3} \Rightarrow x = 2$

39. Official Ans. by NTA (1)

Sol. $y = 4 + \frac{1}{\left(5+\frac{1}{y}\right)}$

$y - 4 = \frac{y}{(5y+1)}$

$5y^2 - 20y - 4 = 0$

$y = \frac{20+\sqrt{480}}{10}$

$y = \frac{20-\sqrt{480}}{10} \rightarrow$ rejected

$y = 2 + \sqrt{\frac{480}{100}}$

Correct with Option (A)

40. Official Ans. by NTA (4)

Sol. $(2023-2)^{3762} = 2023k_1 + 2^{3762}$

$= 17k_2 + 2^{3762}$ (as $2023 = 17 \times 17 \times 9$)

$= 17k_2 + 4 \times 16^{940}$

$= 17k_2 + 4 \times (17-1)^{940}$

$= 17k_2 + 4(17k_3 + 1)$

$= 17k + 4 \Rightarrow$ remainder $= 4$

41. Official Ans. by NTA (2)

Sol. $(1 + x + 2x^2)^{20} = a_0 + a_1x + ... + a_{40}x^{40}$

put $x = 1, -1$

$\Rightarrow a_0 + a_1 + a_2 + ... + a_{40} = 2^{20}$

$a_0 - a_1 + a_2 + ... + a_{40} = 2^{20}$

$\Rightarrow a_1 + a_3 + ... + a_{39} = \dfrac{4^{20} - 2^{20}}{2}$

$\Rightarrow a_1 + a_3 + ... + a_{37} = 2^{39} - 2^{19} - a_{39}$

here $a_{39} = \dfrac{20!(2)^{19} \times 1}{19!} = 20 \times 2^{19}$

$\Rightarrow a_1 + a_3 + ... + a_{37} = 2^{19}(2^{20} - 1 - 20)$

$= 2^{19}(2^{20} - 21)$

42. Official Ans. by NTA (160)

Sol. $\sum_{r=1}^{10} r!\{(r+1)(r+2)(r+3) - 9(r+1) + 8\}$

$= \sum_{r=1}^{10}\left[\{(r+3)! - (r+1)!\} - 8\{(r+1)! - r!\}\right]$

$= (13! + 12! - 2! - 3!) - 8(11! - 1)$

$= (12.13 + 12 - 8).11! - 8 + 8$

$= (160)(11)!$

Hence $\alpha = 160$

43. Official Ans. by NTA (8)

Sol. Let $p'(x) = a(x-1)(x+1) = a(x^2-1)$

$p(x) = a\int (x^2 - 1)\,dx + c$

$= a\left(\dfrac{x^3}{3} - x\right) + c$

Now $p(-3) = 0$

$\Rightarrow a\left(-\dfrac{27}{3} + 3\right) + c = 0$

$\Rightarrow -6a + c = 0$ ...(1)

Now $\int_{-1}^{1}\left(a\left(\dfrac{x^3}{3} - x\right) + c\right)dx = 18$

$= 2c = 18 \Rightarrow c = 9$ ...(2)

$\Rightarrow$ from (1) & (2) $\Rightarrow -6a + 9 = 0 \Rightarrow a = \dfrac{3}{2}$

$\Rightarrow p(x) = \dfrac{3}{2}\left(\dfrac{x^3}{3} - x\right) + 9$

sum of coefficient

$= \dfrac{1}{2} - \dfrac{3}{2} + 9$

$= 8$

44. Official Ans. by NTA (19)

Allen Answer (Bonus)

Sol. BONUS

Instead of ${}^nC_k$ it must be ${}^{10}C_k$ i.e.

$\sum_{k=0}^{10}(2^2 + 3k)\,{}^{10}C_k = \alpha.3^{10} + \beta.2^{10}$

LHS $= 4\sum_{k=0}^{10} {}^{10}C_k + 3\sum_{k=0}^{10} k.\dfrac{10}{k}.{}^{9}C_{k-1}$

$= 4.2^{10} + 3.10.2^9$

$= 19.2^{10} = \alpha.3^{10} + \beta.2^{10}$

$\Rightarrow \alpha = 0, \beta = 19 \Rightarrow \alpha + \beta = 19$