VECTORS

1. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three units vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. If $\lambda = \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$ and $\vec{d}=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$, then the ordered pair, $(\lambda,\vec{d})$ is equal to :

(1) $\left(-\frac{3}{2},3\vec{a}\times\vec{b}\right)$ (2) $\left(-\frac{3}{2},3\vec{c}\times\vec{b}\right)$

(3) $\left(\frac{3}{2},3\vec{b}\times\vec{c}\right)$ (4) $\left(\frac{3}{2},3\vec{a}\times\vec{c}\right)$

2. A vector $\vec{a}=\alpha\hat{i}+2\hat{j}+\beta\hat{k}(\alpha,\beta\in R)$ lies in the plane of the vectros $\vec{b}=\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}-\hat{j}+4\hat{k}$. If $\vec{a}$ bisects the angle between $\vec{b}$ and $\vec{c}$, then:

(1) $\vec{a}.\hat{i}+1=0$ (2) $\vec{a}.\hat{i}+3=0$

(3) $\vec{a}.\hat{k}+4=0$ (4) $\vec{a}.\hat{k}+2=0$

3. Let $\vec{a}=\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ be two vectors. If $\vec{c}$ is a vector such that $\vec{b}\times\vec{c}=\vec{b}\times\vec{a}$ and $\vec{c}\cdot\vec{a}=0$, then $\vec{c}\cdot\vec{b}$ is equal to

(1) $\frac{1}{2}$ (2) -1

(3) $-\frac{1}{2}$ (4) $-\frac{3}{2}$

4. Let the volume of a parallelopiped whose coterminous edges are given by $\vec{u}=\hat{i}+\hat{j}+\lambda\hat{k},\vec{v}=\hat{i}+\hat{j}+3\hat{k}$ and $\vec{w}=2\hat{i}+\hat{j}+\hat{k}$ be 1 cu. unit. If θ be the angle between the edges $\vec{u}$ and $\vec{w}$, then cosθ can be

(1) $\frac{7}{6\sqrt{3}}$ (2) $\frac{5}{7}$

(3) $\frac{7}{6\sqrt{6}}$ (4) $\frac{5}{3\sqrt{3}}$

5. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}|=\sqrt{3}$, $|\vec{b}|=5$, $\vec{b}.\vec{c}=10$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$. If $\vec{a}$ is perpendicular to the vector $\vec{b}\times\vec{c}$, then $|\vec{a}\times(\vec{b}\times\vec{c})|$ is equal to _____.

6. If the vectors, $\vec{p}=(a+1)\hat{i}+a\hat{j}+a\hat{k}$, $\vec{q}=a\hat{i}+(a+1)\hat{j}+a\hat{k}$ and $\vec{r}=a\hat{i}+a\hat{j}+(a+1)\hat{k}$ $(a\in R)$ are coplanar and $3(\vec{p}.\vec{q})^2-\lambda|\vec{r}\times\vec{q}|^2=0$, then the value of λ is _____.

7. The projection of the line segment joining the points (1, –1, 3) and (2, –4, 11) on the line joining the points (–1, 2, 3) and (3, –2, 10) is _____.

8. Let $\vec{a},\vec{b}$ and $\vec{c}$ be three unit vectors such that $|\vec{a}-\vec{b}|^2+|\vec{a}-\vec{c}|^2=8$. Then $|\vec{a}+2\vec{b}|^2+|\vec{a}+2\vec{c}|^2$ is equal to ________.

9. Let the position vectors of points 'A' and 'B' be $\hat{i}+\hat{j}+\hat{k}$ and $2\hat{i}+\hat{j}+3\hat{k}$, respectively. A point 'P' divides the line segment AB internally in the ratio λ : 1 (λ > 0). If O is the origin and $\overrightarrow{OB}\cdot\overrightarrow{OP}-3|\overrightarrow{OA}\times\overrightarrow{OP}|^2=6$, then λ is equal to______.

10. The lines $\vec{r}=(\hat{i}-\hat{j})+\ell(2\hat{i}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j})+m(\hat{i}+\hat{j}-\hat{k})$

(1) Intersect when ℓ = 1 and m = 2

(2) Intersect when ℓ = 2 and m = $\frac{1}{2}$

(3) Do not intersect for any values of ℓ and m

(4) Intersect for all values of ℓ and m

11. Let a plane P contain two lines $\vec{r} = \hat{i} + \lambda(\hat{i} + \hat{j})$, $\lambda \in R$ and $\vec{r} = -\hat{j} + \mu(\hat{j} - \hat{k})$, $\mu \in R$. If Q(α, β, γ) is the foot of the perpendicular drawn from the point M(1, 0, 1) to P, then 3(α + β + γ) equals _______.

12. Let x_0 be the point of local maxima of $f(x) = \vec{a}.(\vec{b} \times \vec{c})$, where $\vec{a} = x\hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = -2\hat{i} + x\hat{j} - \hat{k}$ and $\vec{c} = 7\hat{i} - 2\hat{j} + x\hat{k}$. Then the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ at $x = x_0$ is :

(1) –30 (2) 14

(3) –4 (4) –22

13. If $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$, then the value of $|\hat{i} \times (\vec{a} \times \hat{i})|^2 + |\hat{j} \times (\vec{a} \times \hat{j})|^2 + |\hat{k} \times (\vec{a} \times \hat{k})|^2$ is equal to ____.

14. If the volume of a parallelopiped, whose coterminus edges are given by the vectors $\vec{a} = \hat{i} + \hat{j} + n\hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} - n\hat{k}$ and $\vec{c} = \hat{i} + n\hat{j} + 3\hat{k}$ $(n \geq 0)$, is 158 cu. units, then :

(1) $\vec{a} \cdot \vec{c} = 17$ (2) $\vec{b} \cdot \vec{c} = 10$

(3) n = 7 (4) n = 9

15. Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be such that $|\vec{a}| = 2$, $|\vec{b}| = 4$ and $|\vec{c}| = 4$. If the projection of $\vec{b}$ on $\vec{a}$ is equal to the projection of $\vec{c}$ on $\vec{a}$ and $\vec{b}$ is perpendicular to $\vec{c}$, then the value of $|\vec{a} + \vec{b} - \vec{c}|$ is _________.

16. If $\vec{a}$ and $\vec{b}$ are unit vectors, then the greatest value of $\sqrt{3}|\vec{a} + \vec{b}| + |\vec{a} - \vec{b}|$ is _.

17. If $\vec{x}$ and $\vec{y}$ be two non-zero vectors such that $|\vec{x} + \vec{y}| = |\vec{x}|$ and $2\vec{x} + \lambda\vec{y}$ is perpendicular to $\vec{y}$, then the value of λ is __________.

SOLUTION

1. NTA Ans. (1)

Sol. $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b})+2(\vec{b}\cdot\vec{c})+2(\vec{c}\cdot\vec{a})=0$

$\lambda = \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=\frac{-3}{2}$

$\vec{d} = \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$

$\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

$\Rightarrow \vec{d} = 3(\vec{a}\times\vec{b})$

2. NTA Ans. (4)

ALLEN Ans. (BONUS)

Note: None of the given options matches. So, it should be bonus but NTA did not accept our claim

Sol. $\vec{a}=\lambda\left(\hat{b}+\hat{c}\right)=\lambda\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}+\frac{\hat{i}-\hat{j}+4\hat{k}}{3\sqrt{2}}\right)$

$\vec{a}=\frac{\lambda}{3\sqrt{2}}\left(4\hat{i}+2\hat{j}+4\hat{k}\right)\Rightarrow\frac{\lambda}{3\sqrt{2}}(4\hat{i}+2\hat{j}+4\hat{k})$

$=\alpha\hat{i}+2\hat{j}+\beta\hat{k}$

$\Rightarrow \alpha = 4$ and $\beta = 4$

So, $\vec{a}=4\hat{i}+2\hat{j}+4\hat{k}$

None of the given options is correct

3. NTA Ans. (3)

Sol. $\vec{b}\times\vec{c}-\vec{b}\times\vec{a}=\vec{0}$

$\vec{b}\times\left(\vec{c}-\vec{a}\right)=\vec{0}$

$\vec{b}=\lambda\left(\vec{c}-\vec{a}\right)$ ...(i)

$\vec{a}\cdot\vec{b}=\lambda\left(\vec{a}\cdot\vec{c}-\vec{a}^2\right)$

$4 = \lambda(0-6) \Rightarrow \lambda=\frac{-4}{6}=\frac{-2}{3}$

from (i) $\vec{b}=\frac{-2}{3}(\vec{c}-\vec{a})$

$\vec{c}=\frac{-3}{2}\vec{b}+\vec{a}=\frac{-1}{2}\left(\hat{i}+\hat{j}+\hat{k}\right)$

$\boxed{\vec{b}\cdot\vec{c}=\frac{1}{2}}$ (3) Option

4. NTA Ans. (1)

Sol. $\left\|\begin{matrix}1 & 1 & \lambda\\ 1 & 1 & 3\\ 2 & 1 & 1\end{matrix}\right\|=1\Rightarrow\lambda=2,4$

Now, $\cos\theta=\frac{\vec{u}.\vec{w}}{|\vec{u}||\vec{w}|}$

$=\frac{5}{\sqrt{6}\sqrt{6}}$ or $\frac{7}{\sqrt{6}\sqrt{18}}=\frac{5}{6}$ or $\frac{7}{6\sqrt{3}}$

5. NTA Ans. (30)

Sol. $\vec{b}\,.\,\vec{c}=10 \Rightarrow 5|\vec{c}|\cos\frac{\pi}{3}=10 \Rightarrow |\vec{c}|=4$

$\left|\vec{a}\times\left(\vec{b}\times\vec{c}\right)\right|=|\vec{a}|\,|\vec{b}\times\vec{c}|$

$=\sqrt{3}\,.\,5\,.\,4\,.\,\sin\frac{\pi}{4}=30$

6. NTA Ans. (1.00)

Sol. $\vec{p}=(a+1)\hat{i}+a\hat{j}+a\hat{k}$,

$\vec{q}=a\hat{i}+(a+1)\hat{j}+a\hat{k}$ and

$\vec{r}=a\hat{i}+a\hat{j}+(a+1)\hat{k}$

$\because$ $\vec{p},\vec{q},\vec{r}$ are coplanar

$\Rightarrow [\vec{p}\;\;\vec{q}\;\;\vec{r}]=0$

$\Rightarrow\begin{vmatrix}a+1 & a & a\\ a & a+1 & a\\ a & a & a+1\end{vmatrix}=0$

$\Rightarrow 3a+1=0 \Rightarrow a=-\frac{1}{3}$

$\vec{p}.\vec{q}=-\frac{1}{3},\;\vec{r}.\vec{q}=-\frac{1}{3}$

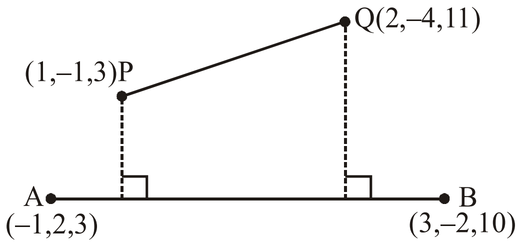
$|\vec{r}|^2=|\vec{q}|^2=\frac{2}{3}$

$\therefore\; 3(\vec{p}.\vec{q})^2-\lambda|\vec{r}\times\vec{q}|^2=0$

$\Rightarrow \lambda=\frac{3(\vec{p}.\vec{q})^2}{|\vec{r}\times\vec{q}|^2}=\frac{3(\vec{p}.\vec{q})^2}{|\vec{r}|^2|\vec{q}|^2-(\vec{r}.\vec{q})^2}=1.00$

7. NTA Ans. (8.00)

Sol.



Projection of $\overline{PQ}$ on $\overline{AB} = \left|\dfrac{\overline{PQ}\cdot\overline{AB}}{|\overline{AB}|}\right|$

$$=\left|\frac{(\hat{i}-3\hat{j}+8\hat{k})\cdot(4\hat{i}-4\hat{j}+7\hat{k})}{9}\right| = 8$$

8. Official Ans. by NTA (2.00)

Sol. $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

$\left|\vec{a}-\vec{b}\right|^2 + \left|\vec{a}-\vec{b}\right|^2 = 8$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b} + |\vec{a}|^2 + |\vec{c}|^2 - 2\vec{a}.\vec{c} = 8$

$\Rightarrow 4 - 2(\vec{a}.\vec{b} + \vec{a}.\vec{c}) = 8$

$\Rightarrow \boxed{\vec{a}.\vec{b} + \vec{a}.\vec{c} = -2}$

$|\vec{a} + 2\vec{b}|^2 + |\vec{a} + 2\vec{c}|^2$

$= |a^2| + 4|\vec{b}|^2 + 4\vec{a}.\vec{b} + |\vec{a}|^2 + 4|\vec{c}|^2 + 4\vec{a}.\vec{c}$

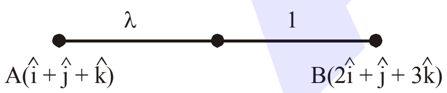
$= 10 + 4(\vec{a}.\vec{b} + \vec{a}.\vec{c})$

$= 10 - 8$

$= \boxed{2}$

9. Official Ans. by NTA (0.8)

Sol.



Using section formula we get

$$\overline{OP} = \frac{2\lambda+1}{\lambda+1}\hat{i} + \frac{\lambda+1}{\lambda+1}\hat{j} + \frac{3\lambda+1}{\lambda+1}\hat{k}$$

Now $\overline{OB}\cdot\overline{OP} = \dfrac{4\lambda+2+\lambda+1+9\lambda+3}{\lambda+1}$

$$= \frac{14\lambda+6}{\lambda+1}$$

$$\overline{OA}\times\overline{OP} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ \frac{2\lambda+1}{\lambda+1} & 1 & \frac{3\lambda+1}{\lambda+1} \end{vmatrix}$$

$$= \frac{2\lambda+1}{\lambda+1}\hat{i} + \frac{-\lambda}{\lambda+1}\hat{j} + \frac{-\lambda}{\lambda+1}\hat{k}$$

$$|\overline{OA}\times\overline{OP}|^2 = \frac{(2\lambda+1)^2+\lambda^2+\lambda^2}{(\lambda+1)^2}$$

$$= \frac{6\lambda^2+1}{(\lambda+1)^2}$$

$$\Rightarrow \frac{14\lambda+6}{\lambda+1} - 3\times\frac{(6\lambda^2+1)}{(\lambda+1)^2} = 6$$

$\Rightarrow 10\lambda^2 - 8\lambda = 0$

$\Rightarrow \lambda = 0, \dfrac{8}{10} = 0.8$

$\Rightarrow \lambda = 0.8$

10. Official Ans. by NTA (3)

Sol. $\vec{r} = \hat{i}(1+2\ell) + \hat{j}(-1) + \hat{k}(\ell)$

$\vec{r} = \hat{i}(2+m) + \hat{j}(m-1) + \hat{k}(-m)$

For intersection

$1 + 2\ell = 2 + m$ ...... (i)

$-1 = m - 1$ ...... (ii)

$\ell = -m$ ...... (iii)

from (ii) $m = 0$

from (iii) $\ell = 0$

These values of m and ℓ do not satisfy equation (1).

Hence the two lines do not intersect for any values of ℓ and m.

11. Official Ans. by NTA (5)

Sol. Dr's normal to plane

$$= \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = -\hat{i} + \hat{j} + \hat{k}$$

Equation of plane

$-1(x-1)+1(y-0)+1(z-0)=0$

$x-y-z-1=0$

......(1)

Now $\frac{\alpha-1}{1}=\frac{\beta-0}{-1}=\frac{\gamma-1}{-1}=-\frac{(1-0-1-1)}{3}$

$\frac{\alpha-1}{1}=\frac{\beta}{-1}=\frac{\gamma-1}{-1}=\frac{1}{3}$

$\alpha=\frac{4}{3},\ \beta=-\frac{1}{3},\ \gamma=\frac{2}{3}$

$3(\alpha+\beta+\gamma)=3\left(\frac{4}{3}-\frac{1}{3}+\frac{2}{3}\right)=5$

12. Official Ans. by NTA (4)

Sol. $f(x)=\vec{a}.\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix} x & -2 & 3 \\ -2 & x & -1 \\ 7 & -2 & x \end{vmatrix}=x^3-27x+26$

$f'(x)=3x^2-27=0 \Rightarrow x=\pm 3$

and $f''(-3)<0$

$\Rightarrow$ local maxima at $x=x_0=-3$

Thus, $\vec{a}=-3\hat{i}-2\hat{j}+3\hat{k}$,

$\vec{b}=-2\hat{i}-3\hat{j}-\hat{k}$,

and $\vec{c}=7\hat{i}-2\hat{j}-3\hat{k}$

$\Rightarrow \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}=9-5-26=-22$

13. Official Ans. by NTA (18)

Sol. $\Sigma|\vec{a}-(\vec{a}\cdot i)i|^2$

$\Rightarrow \Sigma\left(|a|^2+(\vec{a}\cdot i)^2-2(\vec{a}\cdot i)^2\right)$

$\Rightarrow 3|\vec{a}|^2-\Sigma(\vec{a}\cdot i)^2$

$\Rightarrow 2|\vec{a}|^2$

$\Rightarrow 18$

14. Official Ans. by NTA (2)

Sol. $v=\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

$158=\begin{vmatrix} 1 & 1 & n \\ 2 & 4 & -n \\ 1 & n & 3 \end{vmatrix},\ n\geq 0$

$158=1(12+n^2)-(6+n)+n(2n-4)$

$158=n^2+12-6-n+2n^2-4n$

$3n^2-5n-152=0$

$n=8,\ -\frac{38}{6}$ (rejected)

$\vec{a}\cdot\vec{c}=1+n+3n=1+4n=33$

$\vec{b}\cdot\vec{c}=2+4n-3n=2+n=10$

15. Official Ans. by NTA (6.00)

Sol. Projection of $\vec{b}$ on $\vec{a}$ = projection of $\vec{c}$ on $\vec{a}$

$\Rightarrow \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|}=\frac{\vec{c}\cdot\vec{a}}{|\vec{a}|}\Rightarrow \vec{b}\cdot\vec{a}=\vec{c}\cdot\vec{a}$

$\because \vec{b}$ is perpendicular to $\vec{c} \Rightarrow \vec{b}\cdot\vec{c}=0$

Let $|\vec{a}+\vec{b}-\vec{c}|=k$

Square both sides

$k^2=\vec{a}^2+\vec{b}^2+\vec{c}^2+2\vec{a}\cdot\vec{b}-2\vec{a}\cdot\vec{c}-2\vec{b}\cdot\vec{c}$

$k^2=\vec{a}^2+\vec{b}^2+\vec{c}^2=36$

$k=6=|\vec{a}+\vec{b}-\vec{c}|$

16. Official Ans. by NTA (4.00)

Sol. $\sqrt{3}|\vec{a}+\vec{b}|+|\vec{a}-\vec{b}|$

$=\sqrt{3}\left(\sqrt{2+2\cos\theta}\right)+\sqrt{2-2\cos\theta}$

$=\sqrt{6}\left(\sqrt{1+\cos\theta}\right)+\sqrt{2}\left(\sqrt{1-\cos\theta}\right)$

$=2\sqrt{3}\left|\cos\frac{\theta}{2}\right|+2\left|\sin\frac{\theta}{2}\right|$

$\leq\sqrt{\left(2\sqrt{3}\right)^2+(2)^2}=4$

17. Official Ans. by NTA (1.00)

Sol. $|\vec{x}+\vec{y}|=|\vec{x}|$

$\sqrt{|\vec{x}|^2+|\vec{y}|^2+2\vec{x}.\vec{y}}=|\vec{x}|$

$|\vec{y}|^2+2\vec{x}.\vec{y}=0$ .... (1)

Now $(2\vec{x}+\lambda\vec{y}).\vec{y}=0$

$2\vec{x}.\vec{y}+\lambda|\vec{y}|^2=0$

from (1)

$-|\vec{y}|^2+\lambda|\vec{y}|^2=0$

$(\lambda-1)|\vec{y}|^2=0$

given $|\vec{y}|\neq 0 \Rightarrow \lambda=1$