MONOTONICITY

1. The value of c in the Lagrange's mean value theorem for the function $f(x)=x^3-4x^2+8x+11$, when x $\in$ [0, 1] is :

(1) $\frac{2}{3}$ (2) $\frac{\sqrt{7}-2}{3}$

(3) $\frac{4-\sqrt{5}}{3}$ (4) $\frac{4-\sqrt{7}}{3}$

2. Let the function, $f:[-7,0]\to R$ be continuous on [–7, 0] and differentiable on (–7, 0). If $f(-7)=-3$ and $f'(x)\le 2$, for all $x\in(-7,0)$, then for all such functions f, $f(-1)+f(0)$ lies in the interval:

(1) [–6, 20] (2) ($-\infty$, 20]

(3) ($-\infty$, 11] (4) [–3, 11]

3. Let S be the set of all functions $f:[0,1]\to R$, which are continuous on [0,1] and differentiable on (0,1). Then for every f in S, there exists a $c\in(0,1)$, depending on f, such that

(1) $|f(c)-f(1)|<(1-c)|f'(c)|$

(2) $|f(c)-f(1)|<|f'(c)|$

(3) $|f(c)+f(1)|<(1+c)|f'(c)|$

(4) $\frac{f(1)-f(c)}{1-c}=f'(c)$

4. If c is a point at which Rolle's theorem holds for the function, $f(x)=\log_e\left(\frac{x^2+\alpha}{7x}\right)$ in the interval [3,4], where $\alpha\in R$, then $f''(c)$ is equal to

(1) $\frac{\sqrt{3}}{7}$ (2) $\frac{1}{12}$

(3) $-\frac{1}{24}$ (4) $-\frac{1}{12}$

5. Let $f(x)=x\cos^{-1}(-\sin|x|)$, $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then which of the following is true ?

(1) f' is decreasing in $\left(-\frac{\pi}{2},0\right)$ and increasing in $\left(0,\frac{\pi}{2}\right)$

(2) f is not differentiable at x = 0

(3) $f'(0)=-\frac{\pi}{2}$

(4) f' is increasing in $\left(-\frac{\pi}{2},0\right)$ and decreasing in $\left(0,\frac{\pi}{2}\right)$

6. Let f be any function continuous on [a, b] and twice differentiable on (a, b). If for all $x\in(a,b)$, $f'(x)>0$ and $f''(x)<0$, then for any $c\in(a,b)$, $\frac{f(c)-f(a)}{f(b)-f(c)}$ is greater than :

(1) $\frac{b+a}{b-a}$ (2) $\frac{b-c}{c-a}$

(3) $\frac{c-a}{b-c}$ (4) 1

7. Let $f:(-1,\infty)\to R$ be defined by f(0) = 1 and $f(x)=\frac{1}{x}\log_e(1+x), x\ne 0$. Then the function f:

(1) decreases in $(-1,\infty)$

(2) decreases in (–1, 0) and increases in $(0,\infty)$

(3) increases in $(-1,\infty)$

(4) increases in (–1, 0) and decreases in $(0,\infty)$

8. The function, $f(x) = (3x - 7)x^{2/3}$, $x \in R$, is increasing for all x lying in :

(1) $(-\infty, 0) \cup \left(\frac{3}{7}, \infty\right)$

(2) $(-\infty, 0) \cup \left(\frac{14}{15}, \infty\right)$

(3) $\left(-\infty, \frac{14}{15}\right)$

(4) $\left(-\infty, -\frac{14}{15}\right) \cup (0, \infty)$

9. Let f be a twice differentiable function on (1, 6). If $f(2) = 8$, $f'(2) = 5$, $f'(x) \geq 1$ and $f''(x) \geq 4$, for all $x \in (1, 6)$, then :

(1) $f(5) \leq 10$ (2) $f'(5) + f''(5) \leq 20$

(3) $f(5) + f'(5) \geq 28$ (4) $f(5) + f'(5) \leq 26$

10. For all twice differentiable functions $f : R \to R$, with $f(0) = f(1) = f'(0) = 0$

(1) $f''(x) = 0$, for some $x \in (0, 1)$

(2) $f''(0) = 0$

(3) $f''(x) \neq 0$ at every point $x \in (0, 1)$

(4) $f''(x) = 0$ at every point $x \in (0, 1)$

11. If the tangent to the curve, $y = f(x) = x\log_e x$, $(x > 0)$ at a point $(c, f(c))$ is parallel to the line - segement joining the points (1, 0) and (e, e), then c is equal to :

(1) $\frac{1}{e-1}$ (2) $e^{\left(\frac{1}{1-e}\right)}$

(3) $e^{\left(\frac{1}{e-1}\right)}$ (4) $\frac{e-1}{e}$

SOLUTION

1. NTA Ans. (4)

Sol. f(0) = 11

f(1) = 16

$$\frac{f(1)-f(0)}{1-0}=3c^2-8c+8$$

$\Rightarrow$ $3c^2 - 8c + 8 = 5$

$\Rightarrow$ $3c^2 - 8c + 3 = 0$

$$c \in [0, 1] \Rightarrow c = \frac{4-\sqrt{7}}{3}$$

2. NTA Ans. (2)

Sol. Using LMVT in [–7, –1]

$$\frac{f(-1)-f(-7)}{-1-(-7)}\le 2$$

$f(-1) - f(-7) \le 12$

$\Rightarrow f(-1) \le 9$ ....(1)

Using LMVT in [–7, 0]

$$\frac{f(0)-f(-7)}{0-(-7)}\le 2$$

$f(0) - f(-7) \le 14$

$f(0) \le 11$ ....(2)

from (1) and (2)

$f(0) + f(-1) \le 20$

3. NTA Ans. (2)

ALLEN Ans. (BONUS)

Note: None of the options is correct for all *f* in S. Thus, it should be bonus, but NTA did not accept it.

Sol. Option (1), (2), (3) are incorrect for $f(x)$ = constant and option (4) is incorrect

$$\frac{f(1)-f(c)}{1-c}=f'(a) \text{ where } c<a<1 \text{ (use LMVT)}$$

Also for $f(x) = x^2$ option (4) is incorrect.

4. NTA Ans. (2)

Sol. $\frac{9+\alpha}{21}=\frac{16+\alpha}{28}\Rightarrow \alpha=12$

Also, $f'(x)=\frac{7x}{x^2+12}\times\frac{x^2-12}{7x^2}=\frac{x^2-12}{x\left(x^2+12\right)}$

Hence, $c=2\sqrt{3}$

Now, $f''(c)=\frac{1}{12}$

5. NTA Ans. (1)

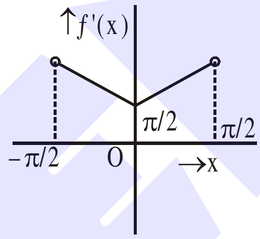
Sol. $f(x)$ is an odd function.

Now, if $x \ge 0$, then $f(x) = x\cos^{-1}(-\sin x)$

$$= x\left(\frac{\pi}{2}-\sin^{-1}(-\sin x)\right)=x\left(\frac{\pi}{2}+x\right)$$

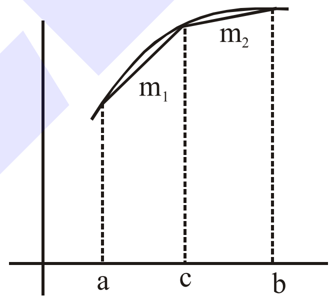
Hence, $f(x)=\begin{cases} x\left(\frac{\pi}{2}+x\right) & ; \quad x\in\left[0,\frac{\pi}{2}\right] \\ x\left(\frac{\pi}{2}-x\right) & ; \quad x\in\left[-\frac{\pi}{2},0\right) \end{cases}$

so, $f'(x)=\begin{cases} \frac{\pi}{2}+2x & ; \quad x\in\left[0,\frac{\pi}{2}\right) \\ \frac{\pi}{2}-2x & ; \quad x\in\left(-\frac{\pi}{2},0\right) \end{cases}$



6. NTA Ans. (3)

Sol.



it is clear from graph that $m_1 > m_2$

$$\Rightarrow \frac{f(c)-f(a)}{c-a}>\frac{f(b)-f(c)}{b-c}$$

$$\Rightarrow \frac{f(c)-f(a)}{f(b)-f(c)}>\frac{c-a}{b-c}$$

7. Official Ans. by NTA (1)

Sol. $f'(x)=\dfrac{\frac{x}{1+x}-\ell n(1+x)}{x^2}$

$$=\frac{x-(1+x)\,\ell n(1+x)}{x^2(1+x)}$$

Suppose $h(x) = x - (1 + x)\ \ell n(1+x)$

$\Rightarrow\ h'(x) = 1 - \ell n(1+x) - 1 = -\ell n(1+x)$

$h'(x) > 0,\ \forall\ x \in (-1, 0)$

$h'(x) < 0,\ \forall\ x \in (0, \infty)$

$h(0) = 0 \Rightarrow h'(x) < 0\ \forall\ x \in (-1, \infty)$

$\Rightarrow f'(x) < 0\ \forall\ x \in (-1, \infty)$

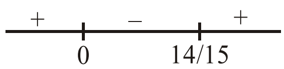
$\Rightarrow$ f(x) is a decreasing function for all $x \in (-1, \infty)$

8. Official Ans. by NTA (2)

Sol. $f(x) = (3x - 7)x^{2/3}$

$\Rightarrow f(x) = 3x^{5/3} - 7x^{2/3}$

$\Rightarrow f'(x) = 5x^{2/3} - \dfrac{14}{3x^{1/3}} = \dfrac{15x - 14}{3x^{1/3}} > 0$



$\therefore\ f'(x) > 0\ \forall\ x \in (-\infty, 0) \cup \left(\dfrac{14}{15}, \infty\right)$

9. Official Ans. by NTA (3)

Sol. $f(2) = 8, f'(2) = 5, f'(x) \geq 1, f''(x) \geq 4, \forall x \in (1,6)$

$f''(x) = \dfrac{f'(5) - f'(2)}{5 - 2} \geq 4 \Rightarrow f'(5) \geq 17 \quad ...(1)$

$f'(x) = \dfrac{f(5) - f(2)}{5 - 2} \geq 1 \Rightarrow f(5) \geq 11 \quad ...(2)$

$\overline{f'(5) + f(5) \geq 28}$

10. Official Ans. by NTA (1)

Sol. $f(0) = f(1) = f'(0) = 0$

Apply Rolles theorem on $y = f(x)$ in $x \in [0, 1]$

$f(0) = f(1) = 0$

$\Rightarrow f'(\alpha) = 0$ where $\alpha \in (0, 1)$

Now apply Rolles theorem on $y = f'(x)$ in $x \in [0, \alpha]$

$f'(0) = f'(\alpha) = 0$ and $f'(x)$ is continuous and differentiable

$\Rightarrow f''(\beta) = 0$ for some , $\beta \in (0, \alpha) \in (0, 1)$

$\Rightarrow f''(x) = 0$ for some $x \in (0, 1)$

11. Official Ans. by NTA (3)

Sol. $f(x) = x\log_e x$

$f'(x)\Big|_{(c, f(c))} = \dfrac{e - 0}{e - 1}$

$f'(x) = 1 + \log_e x$

$f'(x)\Big|_{(c, f(c))} = 1 + \log_e c = \dfrac{e}{e - 1}$

$\log_e c = \dfrac{e - (e - 1)}{e - 1} = \dfrac{1}{e - 1} \Rightarrow c = e^{\frac{1}{e-1}}$