METHOD OF DIFFERENTIATION

1. Let y = y(x) be a function of x satisfying $y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$ where k is a constant and $y\left(\frac{1}{2}\right) = -\frac{1}{4}$. Then $\frac{dy}{dx}$ at $x = \frac{1}{2}$, is equal to:

(1) $\frac{\sqrt{5}}{2}$ (2) $-\frac{\sqrt{5}}{2}$

(3) $\frac{2}{\sqrt{5}}$ (4) $-\frac{\sqrt{5}}{4}$

2. If $y(\alpha) = \sqrt{2\left(\frac{\tan\alpha + \cot\alpha}{1+\tan^2\alpha}\right) + \frac{1}{\sin^2\alpha}}, \alpha \in \left(\frac{3\pi}{4}, \pi\right)$, then $\frac{dy}{d\alpha}$ at $\alpha = \frac{5\pi}{6}$ is :

(1) 4 (2) $-\frac{1}{4}$

(3) $\frac{4}{3}$ (4) –4

3. Let $x^k + y^k = a^k$, (a, K > 0) and $\frac{dy}{dx} + \left(\frac{y}{x}\right)^{\frac{1}{3}} = 0$, then k is :

(1) $\frac{3}{2}$ (2) $\frac{1}{3}$

(3) $\frac{2}{3}$ (4) $\frac{4}{3}$

4. Let $f(x) = (\sin(\tan^{-1}x) + \sin(\cot^{-1}x))^2 - 1$, $|x| > 1$. If $\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\left(\sin^{-1}(f(x))\right)$ and $y(\sqrt{3}) = \frac{\pi}{6}$, then $y(-\sqrt{3})$ is equal to

(1) $\frac{5\pi}{6}$ (2) $-\frac{\pi}{6}$

(3) $\frac{\pi}{3}$ (4) $\frac{2\pi}{3}$

5. If x = 2sinθ – sin2θ and y = 2cosθ – cos2θ, θ ∈ [0, 2π], then $\frac{d^2y}{dx^2}$ at θ = π is :

(1) $\frac{3}{2}$ (2) $-\frac{3}{4}$

(3) $\frac{3}{4}$ (4) $-\frac{3}{8}$

6. Let f and g be differentiable functions on **R** such that fog is the identity function. If for some a, b ∈ **R**, g'(a) = 5 and g(a) = b, then $f'(b)$ is equal to :

(1) $\frac{2}{5}$ (2) 1

(3) $\frac{1}{5}$ (4) 5

7. If $y = \sum_{k=1}^{6} k\cos^{-1}\left\{\frac{3}{5}\cos kx - \frac{4}{5}\sin kx\right\}$, then $\frac{dy}{dx}$ at x = 0 is________.

8. If $y^2 + \log_e(\cos^2 x) = y$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then :

(1) |y''(0)| = 2

(2) |y'(0)| + |y''(0)| = 3

(3) |y'(0)| + |y''(0)| = 1

(4) y''(0) = 0

9. If $\left(a+\sqrt{2}\,b\cos x\right)\left(a-\sqrt{2}\,b\cos y\right) = a^2 - b^2$, where a > b > 0, then $\frac{dx}{dy}$ at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ is :

(1) $\frac{a-b}{a+b}$ (2) $\frac{a+b}{a-b}$

(3) $\frac{2a+b}{2a-b}$ (4) $\frac{a-2b}{a+2b}$

10. The derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = \frac{1}{2}$ is :

(1) $\frac{\sqrt{3}}{12}$ (2) $\frac{\sqrt{3}}{10}$

(3) $\frac{2\sqrt{3}}{5}$ (4) $\frac{2\sqrt{3}}{3}$

SOLUTION

1. NTA Ans. (2)

Sol. Put x = sinθ, y = sinα

$$y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$$

$\Rightarrow$ sinα · cosθ + cosα · sinθ = k

$\Rightarrow$ sin(α + θ) = k

$\Rightarrow$ α + θ = $\sin^{-1}k$

$\Rightarrow$ $\sin^{-1}x + \sin^{-1}y = \sin^{-1}k$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \times \frac{dy}{dx} = 0$$

at $x = \frac{1}{2}$, $y = \frac{-1}{4}$

$$\frac{dy}{dx} = \frac{-\sqrt{5}}{2}$$

2. NTA Ans. (1)

Sol. $y(\alpha) = \sqrt{2\frac{(\tan\alpha + \cot\alpha)}{1+\tan^2\alpha} + \frac{1}{\sin^2\alpha}}$, $\alpha \in \left(\frac{3\pi}{4}, \pi\right)$

$$= \frac{|\sin\alpha + \cos\alpha|}{|\sin\alpha|} = \frac{-(\sin\alpha + \cos\alpha)}{\sin\alpha}$$

= – 1 – cotα

y' (α) = $\operatorname{cosec}^2\alpha$

$$y'\left(\frac{5\pi}{6}\right) = 4$$

3. NTA Ans. (3)

Sol. $x^k + y^k = a^k$ (a ,k > 0)

$$kx^{k-1} + ky^{k-1}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} + \left(\frac{x}{y}\right)^{k-1} = 0 \Rightarrow k - 1 = -\frac{1}{3} \Rightarrow k = 2/3$$

4. NTA Ans. (1)

ALLEN Ans. (BONUS)

Note: The given information is insufficient to find y(x) for x < –1. So, it should be bonus, but NTA retained its answer as options.

Sol. Let $\tan^{-1}x = \theta$, $\theta \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$f(x) = (\sin\theta + \cos\theta)^2 - 1 = \sin 2\theta = \frac{2x}{1+x^2}$$

Now, $\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$= -\frac{1}{1+x^2}, |x| > 1$$

Since, we can integrate only in the continuous interval. So we have to take integral in two cases separtely namely for x < –1 and for x > 1.

$$\Rightarrow y = \begin{cases} -\tan^{-1}x + c_1 & ; \quad x > 1 \\ -\tan^{-1}x + c_2 & ; \quad x < -1 \end{cases}$$

so, $c_1 = \frac{\pi}{2}$ as $y(\sqrt{3}) = \frac{\pi}{6}$

But we cannot find c_2 as we do not have any other additional information for x < –1. So, all of the given options may be correct as c_2 is unknown so, it should be bonus.

5. NTA Ans. (BONUS)

Note: This question has been cancelled by NTA as none option matches.

Sol. x = 2sinθ – sin2θ

$$\Rightarrow \frac{dx}{d\theta} = 2\cos\theta - 2\cos 2\theta = 4\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)$$

y = 2cosθ – cos2θ

$$\Rightarrow \frac{dy}{d\theta} = -2\sin\theta + 2\sin 2\theta = 4\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\Rightarrow \frac{dy}{dx} = \cot\left(\frac{3\theta}{2}\right) \Rightarrow \frac{d^2y}{dx^2} = \frac{-\frac{3}{2}\operatorname{cosec}^2\left(\frac{3\theta}{2}\right)}{4\sin\left(\frac{\theta}{2}\right)\sin\frac{3\theta}{2}}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\theta=\pi} = \frac{3}{8}$$

Alternate :-

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{-2\sin\theta+2\sin2\theta}{2\cos\theta-2\cos2\theta}=\frac{\sin\theta-\sin2\theta}{-\cos\theta+\cos2\theta}$$

$$\frac{d^2y}{dx^2}.\frac{dx}{d\theta}=$$

$$\frac{(-\cos\theta+\cos2\theta)(\cos\theta-2\cos2\theta)-(\sin\theta-\sin2\theta)(\sin\theta-2\sin2\theta)}{(-\cos\theta+\cos2\theta)^2}$$

$$\frac{d^2y}{dx^2}.(-2-2)=\frac{(+1+1)(-1-2)-(0)}{(1+1)^2}$$

$$\frac{d^2y}{dx^2}(-4)=\frac{2\times-3}{4}=-\frac{3}{2}$$

$$\frac{d^2y}{dx^2}=\frac{3}{8}$$

Answer should be $\frac{3}{8}$. No options is correct.

6. NTA Ans. (3)

Sol. $f(g(x)) = x$

$f'(g(x))\, g'(x) = 1$

put x = a

$\Rightarrow f'(b)\, g'(a) = 1$

$f'(b) = \frac{1}{5}$

7. Official Ans. by NTA (91)

Sol. Put $\cos\alpha=\frac{3}{5},\sin\alpha=\frac{4}{5}\quad 0<\alpha<\frac{\pi}{2}$

Now $\frac{3}{5}\cos kx-\frac{4}{5}\sin kx$

$= \cos\alpha . \cos kx - \sin\alpha . \sin kx$

$= \cos(\alpha + kx)$

As we have to find derivate at x = 0

We have $\cos^{-1}(\cos(\alpha + kx))$

$= (\alpha + kx)$

$$\Rightarrow y=\sum_{k=1}^{6}(\alpha+kx)$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{at\ x=0}=\sum_{k=x}^{6}k=\frac{6\times7\times13}{6}=91$$

8. Official Ans. by NTA (1)

Sol. $y^2 + \ln(\cos^2 x) = y \quad x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

for x = 0 y = 0 or 1

Differentiating wrt x

$\Rightarrow 2yy' - 2\tan x = y'$

At (0, 0) y' = 0

At (0, 1) y' = 0

Differentiating wrt x

$2yy'' + 2(y')^2 - 2\sec^2 x = y''$

At (0, 0) y'' = –2

At (0, 1) y'' = 2

$\therefore |y''(0)| = 2$

9. Official Ans. by NTA (2)

Sol. $\left(a+\sqrt{2}b\cos x\right)\left(a-\sqrt{2}b\cos y\right)=a^2-b^2$

$$\Rightarrow a^2-\sqrt{2}\,ab\cos y+\sqrt{2}\,ab\cos x-2b^2\cos x\cos y=a^2-b^2$$

Differentiating both sides :

$$0-\sqrt{2}\,ab\left(-\sin y\frac{dy}{dx}\right)+\sqrt{2}\,ab(-\sin x)$$

$$-2b^2\left[\cos x\left(-\sin y\frac{dy}{dx}\right)+\cos y(-\sin x)\right]=0$$

At $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$:

$$ab\frac{dy}{dx}-ab-2b^2\left(-\frac{1}{2}\frac{dy}{dx}-\frac{1}{2}\right)=0$$

$$\Rightarrow \frac{dx}{dy}=\frac{ab+b^2}{ab-b^2}=\frac{a+b}{a-b}\ ;\ a,b>0$$

10. Official Ans. by NTA (2)

Sol. Let $f=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$f=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$f=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)=\frac{\theta}{2}$$

$f = \frac{\tan^{-1} x}{2} \Rightarrow \frac{df}{dx} = \frac{1}{2(1+x^2)}$ ....(i)

Let $g = \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

$g = \tan^{-1}\left(\frac{2\sin\theta\cos\theta}{1-2\sin^2\theta}\right)$

$g = \tan^{-1}(\tan 2\theta) = 2\theta$

$g = 2\sin^{-1} x$

$\frac{dg}{dx} = \frac{2}{\sqrt{1-x^2}}$ ....(ii)

$\frac{df}{dg} = \frac{1}{2(1+x^2)} \frac{\sqrt{1-x^2}}{2}$

at $x = \frac{1}{2}$ $\left(\frac{df}{dg}\right)_{x=\frac{1}{2}} = \frac{\sqrt{3}}{10}$