COMPLEX NUMBER

1. If $\frac{3+i\sin\theta}{4-i\cos\theta}, \theta\in[0,2\pi]$, is a real number, then an argument of $\sin\theta + i\cos\theta$ is :

(1) $-\tan^{-1}\left(\frac{3}{4}\right)$ (2) $\tan^{-1}\left(\frac{4}{3}\right)$

(3) $\pi-\tan^{-1}\left(\frac{4}{3}\right)$ (4) $\pi-\tan^{-1}\left(\frac{3}{4}\right)$

2. If $\text{Re}\left(\frac{z-1}{2z+i}\right) = 1$, where $z = x + iy$, then the point (x,y) lies on a :

(1) circle whose centre is at $\left(-\frac{1}{2},-\frac{3}{2}\right)$

(2) circle whose diameter is $\frac{\sqrt{5}}{2}$

(3) straight line whose slope is $\frac{3}{2}$

(4) straight line whose slope is $-\frac{2}{3}$

3. Let $\alpha=\frac{-1+i\sqrt{3}}{2}$. If $a=(1+\alpha)\sum_{k=0}^{100}\alpha^{2k}$ and $b=\sum_{k=0}^{100}\alpha^{3k}$, then a and b are the roots of the quadratic equation :

(1) $x^2 - 102x + 101 = 0$

(2) $x^2 + 101x + 100 = 0$

(3) $x^2 - 101x + 100 = 0$

(4) $x^2 + 102x + 101 = 0$

4. If the equation, $x^2 + bx + 45 = 0$ $(b \in R)$ has conjugate complex roots and they satisfy $|z+1|=2\sqrt{10}$, then

(1) $b^2 - b = 42$ (2) $b^2 + b = 12$

(3) $b^2 + b = 72$ (4) $b^2 - b = 30$

5. If z be a complex number satisfying $|\text{Re}(z)| + |\text{Im}(z)| = 4$, then $|z|$ cannot be

(1) $\sqrt{\frac{17}{2}}$ (2) $\sqrt{10}$

(3) $\sqrt{8}$ (4) $\sqrt{7}$

6. Let z be complex number such that $\left|\frac{z-i}{z+2i}\right|=1$ and $|z|=\frac{5}{2}$. Then the value of $|z + 3i|$ is :

(1) $\sqrt{10}$ (2) $2\sqrt{3}$

(3) $\frac{7}{2}$ (4) $\frac{15}{4}$

7. The value of $\left(\frac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)^3$ is :

(1) $\frac{1}{2}\left(\sqrt{3}-i\right)$ (2) $-\frac{1}{2}\left(\sqrt{3}-i\right)$

(3) $-\frac{1}{2}\left(1-i\sqrt{3}\right)$ (4) $\frac{1}{2}\left(1-i\sqrt{3}\right)$

8. The imaginary part of $\left(3+2\sqrt{-54}\right)^{1/2}$ z $-\left(3-2\sqrt{-54}\right)^{1/2}$ can be :

(1) $-2\sqrt{6}$ (2) 6

(3) $\sqrt{6}$ (4) $-\sqrt{6}$

9. If $\left(\frac{1+i}{1-i}\right)^{\frac{m}{2}}=\left(\frac{1+i}{i-1}\right)^{\frac{n}{3}}=1$, $(m, n \in N)$ then the greatest common divisor of the least values of m and n is _______ .

10. If z_1, z_2 are complex numbers such that $Re(z_1) = |z_1 - 1|$, $Re(z_2) = |z_2 - 1|$ and $\arg(z_1 - z_2) = \frac{\pi}{6}$, then $Im(z_1 + z_2)$ is equal to :

(1) $\frac{\sqrt{3}}{2}$ (2) $\frac{2}{\sqrt{3}}$

(3) $\frac{1}{\sqrt{3}}$ (4) $2\sqrt{3}$

11. If $A = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$, $\left(\theta = \frac{\pi}{24}\right)$ and $A^5 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $i = \sqrt{-1}$, then which one of the following is not true?

(1) $0 \le a^2 + b^2 \le 1$ (2) $a^2 - d^2 = 0$

(3) $a^2 - b^2 = \frac{1}{2}$ (4) $a^2 - c^2 = 1$

12. Let $u = \frac{2z+i}{z-ki}$, $z = x + iy$ and $k > 0$. If the curve represented by $Re(u) + Im(u) = 1$ intersects the y-axis at the points P and Q where PQ = 5, then the value of k is :

(1) 3/2 (2) 4

(3) 2 (4) 1/2

13. If a and b are real numbers such that $(2 + \alpha)^4 = a + b\alpha$, where $\alpha = \frac{-1+i\sqrt{3}}{2}$, then a + b is equal to:

(1) 57 (2) 33

(3) 24 (4) 9

14. If the four complex numbers z, $\bar{z}$, $\bar{z} - 2Re(\bar{z})$ and $z - 2Re(z)$ represent the vertices of a square of side 4 units in the Argand plane, then $|z|$ is equal to :

(1) 4 (2) 2

(3) $4\sqrt{2}$ (4) $2\sqrt{2}$

15. The value of $\left(\frac{-1+i\sqrt{3}}{1-i}\right)^{30}$ is :

(1) $2^{15} i$ (2) -2^{15}

(3) $-2^{15} i$ (4) 6^5

16. The region represented by $\{z = x + iy \in C : |z| - Re(z) \le 1\}$ is also given by the inequality:

(1) $y^2 \ge x + 1$ (2) $y^2 \ge 2(x + 1)$

(3) $y^2 \le x + \frac{1}{2}$ (4) $y^2 \le 2\left(x + \frac{1}{2}\right)$

17. Let $z = x + iy$ be a non-zero complex number such that $z^2 = i|z|^2$, where $i = \sqrt{-1}$, then z lies on the:

(1) imaginary axis

(2) real axis

(3) line, y = x

(4) line, y = –x

SOLUTION

1. NTA Ans. (3)

Sol. $\dfrac{3+i\sin\theta}{4-i\cos\theta}$ is a real number

$\Rightarrow 3\cos\theta + 4\sin\theta = 0$

$\Rightarrow \tan\theta = \dfrac{-3}{4}$

argument of $\sin\theta + i\cos\theta = \pi - \tan^{-1}\dfrac{4}{3}$

2. NTA Ans. (2)

Sol. $\text{Re}\left(\dfrac{z-1}{2z+i}\right)=1$

Put $z = x + iy$

$$\text{Re}\left(\frac{(x+iy)-1}{2(x+iy)+i}\right)=1$$

$$\text{Re}\left(\left(\frac{(x-1)+iy}{2x+i(2y+1)}\right)\left(\frac{2x-i(2y+1)}{2x-i(2y+1)}\right)\right)=1$$

$\Rightarrow 2x^2 + 2y^2 + 2x + 3y + 1 = 0$

$x^2 + y^2 + x + \dfrac{3}{2}y + \dfrac{1}{2} = 0$

$\Rightarrow$ locus is a circle whose

Centre is $\left(-\dfrac{1}{2},-\dfrac{3}{4}\right)$ and radius $\dfrac{\sqrt{5}}{4}$

$\Rightarrow$ diameter $= \dfrac{\sqrt{5}}{2}$

3. NTA Ans. (1)

Sol. $\alpha = \omega$

$a = (1 + \omega)(1 + \omega^2 + \omega^4 + ... + \omega^{200})$

$$a=(1+\omega)\frac{\left(1-\left(\omega^2\right)^{101}\right)}{1-\omega^2}=1$$

$b = 1 + \omega^3 + \omega^6 + ... + \omega^{300} = 101$

$x^2 - 102x + 101 = 0$

(1) Option

4. NTA Ans. (4)

Sol. Assuming z is a root of the given equation,

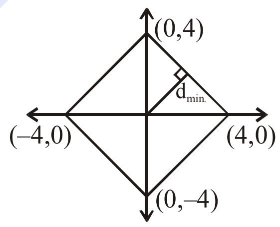
$$z=\frac{-b\pm i\sqrt{180-b^2}}{2}$$

$$\text{so,}\left(1-\frac{b}{2}\right)^2+\frac{180-b^2}{4}=40$$

$\Rightarrow -4b + 184 = 160 \Rightarrow b = 6$

5. NTA Ans. (4)

Sol. $z = x + iy$



$|x| + |y| = 4$

$|z|=\sqrt{x^2+y^2} \Rightarrow |z|_{min}=\sqrt{8}$ & $|z|_{max}=4=\sqrt{16}$

So $|z|$ cannot be $\sqrt{7}$

6. NTA Ans. (3)

Sol. $\left|\frac{z-i}{z+2i}\right|=1$

$\Rightarrow |z - i| = |z + 2i|$

$\Rightarrow$ z lies on perpendicular bisector of (0, 1) and (0, –2).

$\Rightarrow \text{Im}z=-\frac{1}{2}$

Let $z=x-\frac{i}{2}$

$\because |z| = \frac{5}{2} \Rightarrow x^2=6$

$\therefore |z + 3i| =\left|x+\frac{5i}{2}\right|=\sqrt{x^2+\frac{25}{4}}$

$=\sqrt{6+\frac{25}{4}}=\frac{7}{2}$

7. Official Ans. by NTA (2)

Sol. The value of $\left(\frac{1+\sin 2\pi/9+i\cos 2\pi/9}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)$

$=\left(\frac{1+\sin\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)+i\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)}{1+\sin\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)-i\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)}\right)^3$

$=\left(\frac{1+\cos\frac{5\pi}{18}+i\sin\frac{5\pi}{18}}{1+\cos\frac{5\pi}{18}-i\sin\frac{5\pi}{18}}\right)^3$

$=\left(\frac{2\cos^2\frac{5\pi}{36}+2i\sin\frac{5\pi}{36}\cos\frac{5\pi}{36}}{2\cos^2\frac{5\pi}{36}-2i\sin\frac{5\pi}{36}.\cos\frac{5\pi}{36}}\right)^3$

$=\left(\frac{\cos\frac{5\pi}{36}+i\sin\frac{5\pi}{36}}{\cos\frac{5\pi}{36}-i\sin\frac{5\pi}{36}}\right)^3$

$=\left(\frac{e^{i5\pi/36}}{e^{-i5\pi/36}}\right)^3=\left(e^{i5\pi/18}\right)^3$

$=\cos\frac{5\pi}{6}+i\sin 5\pi/6$

$=-\frac{\sqrt{3}}{2}+i/2$

8. Official Ans. by NTA (1)

Sol. $\left(3+2\sqrt{-54}\right)=3+2\times3\times\sqrt{6}\,i$

$=\left(3+\sqrt{6}\,i\right)^2$

$\left(3-2\sqrt{54}\right)=\left(3-\sqrt{6}\,i\right)^2$

$\left(3+2\sqrt{-54}\right)^{1/2}+\left(3-2\sqrt{-54}\right)^{1/2}$

$=\pm\left(3+\sqrt{6}\,i\right)\pm\left(3-\sqrt{6}\,i\right)$

$= 6, -6, 2\sqrt{6}i, -2\sqrt{6}i,$

9. Official Ans. by NTA (4)

Sol. $\left(\frac{1+i}{1-i}\right)^{m/2} = \left(\frac{1+i}{i-1}\right)^{n/3} = 1$

$\Rightarrow \left(\frac{(1+i)^2}{2}\right)^{m/2} = \left(\frac{(1+i)^2}{-2}\right)^{n/3} = 1$

$\Rightarrow (i)^{m/2} = (-i)^{n/3} = 1$

$\Rightarrow \frac{m}{2} = 4k_1$ and $\frac{n}{3} = 4k_2$

$\Rightarrow m = 8k_1$ and $n = 12k_2$

Least value of m = 8 and n = 12.

$\therefore$ GCD = 4

10. Official Ans. by NTA (4)

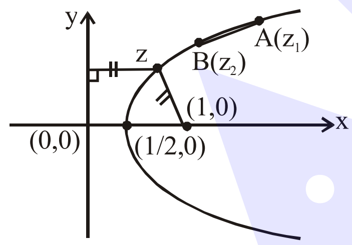
Sol. $\mathrm{Re}(z) = |z-1|$

$\Rightarrow x = \sqrt{(x-1)^2 + (y-0)^2} \qquad (x > 0)$

$\Rightarrow y^2 = 2x - 1 = 4.\frac{1}{2}\left(x - \frac{1}{2}\right)$

$\Rightarrow$ a parabola with focus (1, 0) & directrix as imaginary axis.

$\therefore$ Vertex $= \left(\frac{1}{2}, 0\right)$



$A(z_1)$ & $B(z_2)$ are two points on it such that

slope of AB = $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\left(\arg(z_1 - z_2) = \frac{\pi}{6}\right)$

for $y^2 = 4ax$

Let $A(at_1^2, 2at_1)$ & $B(at_2^2, 2at_2)$

$m_{AB} = \frac{2}{t_1 + t_2} = \frac{4a}{y_1 + y_2} = \frac{1}{\sqrt{3}}$

$\left(\text{Here } a = \frac{1}{2}\right)$

$\Rightarrow y_1 + y_2 = 4a\sqrt{3} = 2\sqrt{3}$

11. Official Ans. by NTA (3)

Sol. $A^2 = \begin{pmatrix} \cos 2\theta & i\sin 2\theta \\ i\sin 2\theta & \cos 2\theta \end{pmatrix}$

Similarly, $A^5 = \begin{pmatrix} \cos 5\theta & i\sin 5\theta \\ i\sin 5\theta & \cos 5\theta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(1) $a^2 + b^2 = \cos^2 5\theta - \sin^2 5\theta = \cos 10\theta = \cos 75°$

(2) $a^2 - d^2 = \cos^2 5\theta - \cos^2 5\theta = 0$

(3) $a^2 - b^2 = \cos^2 5\theta + \sin^2 5\theta = 1$

(4) $a^2 - c^2 = \cos^2 5\theta + \sin^2 5\theta = 1$

12. Official Ans. by NTA (3)

Sol. $u = \frac{2z+i}{z-ki}$

$= \frac{2x^2 + (2y+1)(y-k)}{x^2 + (y-k)^2} + i\frac{(x(2y+1) - 2x(y-k))}{x^2 + (y-k)^2}$

Since Re(u) + Im(u) = 1

$\Rightarrow 2x^2 + (2y+1)(y-k) + x(2y+1) - 2x(y-k) = x^2 + (y-k)^2$

$\left.\begin{matrix} P(0, y_1) \\ Q(0, y_2) \end{matrix}\right\rangle \Rightarrow y^2 + y - k - k^2 = 0 \left\langle \begin{matrix} y_1 + y_2 = -1 \\ y_1 y_2 = -k - k^2 \end{matrix}\right.$

$\because$ PQ = 5

$\Rightarrow |y_1 - y_2| = 5 \Rightarrow k^2 + k - 6 = 0$

$\Rightarrow k = -3, 2$

So, k = 2 (k > 0)

13. Official Ans. by NTA (4)

Sol. $\alpha = \omega \quad (\omega^3 = 1)$

$\Rightarrow (2 + \omega)^4 = a + b\omega$

$\Rightarrow 2^4 + 4.2^3\omega + 6.2^2\omega^3 + 4.2 . \omega^3 + \omega^4 = a + b\omega$

$\Rightarrow 16 + 32\omega + 24\omega^2 + 8 + \omega = a + b\omega$

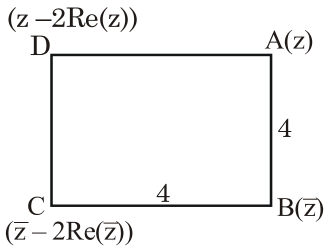
$\Rightarrow 24 + 24\omega^2 + 33\omega = a + b\omega$

$\Rightarrow -24\omega + 33\omega = a + b\omega$

$\Rightarrow a = 0, b = 9$

14. Official Ans. by NTA (4)

Sol. Let z = x + iy



Length of side = 4

AB = 4

$|z - \bar{z}| = 4$

$|2y| = 4$; $|y| = 2$

BC = 4

$\left|\bar{z} - (\bar{z} - 2\,\mathrm{Re}(\bar{z})\right| = 4$

$|2x| = 4$; $|x| = 2$

$|z| = \sqrt{x^2 + y^2} = \sqrt{4+4} = 2\sqrt{2}$

15. Official Ans. by NTA (3)

Sol. $\left(\frac{-1 + i\sqrt{3}}{1-i}\right)^{30} = \left(\frac{2\omega}{1-i}\right)^{30}$

$= \frac{2^{30} \cdot \omega^{30}}{\left((1-i)^2\right)^{30}}$

$= \frac{2^{30} \cdot 1}{(1 + i^2 - 2i)^{15}}$

$= \frac{2^{30}}{-2^{15} \cdot i^{15}}$

$= -2^{15} i$

16. Official Ans. by NTA (4)

Sol. z = x + iy

$|z| - ke(z) \le 1$

$\Rightarrow \sqrt{x^2 + y^2} - x \le 1$

$\Rightarrow \sqrt{x^2 + y^2} \le 1 + x$

$\Rightarrow x^2 + y^2 \le 1 + 2x + x^2$

$\Rightarrow y^2 \le 2x + 1$

$\Rightarrow y^2 \le 2\left(x + \frac{1}{2}\right)$

17. Official Ans. by NTA (3)

Sol. z = x + iy

$z^2 = i|z|^2$

$(x + iy)^2 = i(x^2 + y^2)$

$(x^2 - y^2) - i(x^2 + y^2 - 2xy) = 0$

$(x - y)(x + y) - i(x - y)^2 = 0$

$(x - y)((x + y) - i(x - y)) = 0$

$\Rightarrow x = y$

z lies on y = x