SCORE JEE (Advanced)
HOME ASSIGNMENT # 01

SOLUTION **MATHEMATICS**

1. $f(m+1) - f(m) = 3(4m-1)$

$f(2) - f(1) = 3(4-1)$

$f(3) - f(2) = 3(8-1)$

$f(19) - f(18) = 3(4.18-1)$

$f(19) = 3\,[4(9.19) - 18] = 3.18[37] = 1998$

3. $f_1(x) = f_0 \circ f_0(x) = f_0(f_0(x))$

$$= \frac{f_0(x)}{1+f_0(x)} = \frac{\frac{x}{x+1}}{1+\frac{x}{x+1}} = \frac{x}{1+2x}$$

$f_2(x) = f_0 \circ f_1(x)$

$$= \frac{f_1(x)}{1+f_1(x)} = \frac{\frac{x}{1+2x}}{1+\frac{x}{1+2x}} = \frac{x}{1+3x} \text{ and so on}$$

$$f_n(x) = \frac{x}{1+(n+1)x}$$

5. $x^2 f(x) - 2f\left(\frac{1}{x}\right) = g(x)$ .... (i)

$\Rightarrow$ $x^2 f(-x) - 2f\left(-\frac{1}{x}\right) = g(-x)$

$x^2 f(x) - 2f\left(\frac{1}{x}\right) = -g(x)$ .... (ii)

$2x^2 f(x) - 4f\left(\frac{1}{x}\right) = 0$ {(i) + (ii)}

$x^2 f(x) - 2f\left(\frac{1}{x}\right) = 0$ .... (iii)

Replacing x by 1/x

$\frac{1}{x^2} f\left(\frac{1}{x}\right) - 2f(x) = 0$ .... (iv)

Putting value of f(1/x) from (iii) in (iv)

$\Rightarrow$ $\frac{1}{x^2}.\frac{x^2}{2}.f(x) - 2f(x) = 0 \Rightarrow$ $f(x) = 0$

6. Replacing x by x + 1

$$f(x+2) = \frac{f(x+1)-1}{f(x+1)+1}$$

$$= \frac{\frac{f(x)-1}{f(x)+1}-1}{\frac{f(x)-1}{f(x)+1}+1} = \frac{-2}{2f(x)} = \frac{-1}{f(x)}$$

Replacing x by x + 2

$$f(x+3) = \frac{f(x+2)-1}{f(x+2)+1} = \frac{-\frac{1}{f(x)}-1}{-\frac{1}{f(x)}+1} = \frac{1+f(x)}{1-f(x)}$$

Replacing x by x + 3

$$f(x+4) = \frac{f(x+3)-1}{f(x+3)+1} = \frac{\frac{1+f(x)}{1-f(x)}-1}{\frac{1+f(x)}{1-f(x)}+1}$$

$$= \frac{2f(x)}{2} = f(x)$$

$\therefore$ Function $f(x)$ is periodic with period 4

$\therefore$ $f(999) = f(4 \times 249 + 3) = f(3) = -\frac{1}{2}$

$\{\because$ $f(2) = \frac{f(1)-1}{f(1)+1} = \frac{1}{3}$,

$$f(3) = \frac{f(2)-1}{f(2)+1} = \frac{\frac{1}{3}-1}{\frac{1}{3}+1} = -\frac{1}{2}\ \}$$

9. $$\lim_{x\to 0} \frac{(2^{\sin x}-1)(\ell n(1+\sin 2x))}{x^2 \frac{\tan^{-1}x}{x}}$$

$$= \lim_{x\to 0} \frac{(2^{\sin x}-1)}{\sin x} \times \frac{\sin x}{x} \times \frac{\ell n(1+\sin 2x)}{\sin 2x} \times \frac{\sin 2x}{2x} \times 2$$

$$= \ell n 2 \lim_{x\to 0} \frac{\sin x}{x}.\frac{\sin 2x}{2x} \times 2 = 2\ell n 2$$

12. $y = \lim_{x\to\infty} \left(3^x + 3^{2x}\right)^{1/x}$

$$= 9 \lim_{x\to\infty} \left(1+\frac{1}{3^x}\right)^{1/x} = 9\left\{\lim_{x\to\infty}\left(1+\frac{1}{3^x}\right)^{3^x}\right\}^{\frac{1}{x3^x}} = 9.$$

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13. $s=\cos\left[\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right]+\cos\left[\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right]$

$s = 2\ .\ \cos\frac{\pi}{4}\cos\left[\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right]=\sqrt{2}\ .\ \cos\frac{\theta}{2}$

where $\theta = \cos^{-1}\frac{a}{b} \Rightarrow \quad 0 \le \theta \le \pi$

$\Rightarrow \cos\theta = \frac{a}{b}$

$\Rightarrow \quad 2\cos^2\frac{\theta}{2}=\frac{a+b}{b}$

$\Rightarrow \cos\frac{\theta}{2}=\sqrt{\frac{a+b}{2b}}\ \left\{0\le\frac{\theta}{2}\le\frac{\pi}{2}\right\}$

$\therefore \quad s=\sqrt{\frac{a+b}{b}}$.

15. $100x - 100[x] = 1 \Rightarrow 100(x - [x]) = 1$

$x - [x] = \frac{1}{100}$

$\{x\} = \frac{1}{100} \Rightarrow x = n + \frac{1}{100}, n \in I.$

16. $f(x) = 5\log_5 x$

$y = 5\log_5 x$

$y/5 = \log_5 x$

$x = (5)^{y/5}$

$f^{-1}(x) = 5^{x/5}$

$f^{-1}(\alpha-\beta)=5^{\frac{\alpha-\beta}{5}}=\frac{5^{\alpha/5}}{5^{\beta/5}}=\frac{f^{-1}(\alpha)}{f^{-1}(\beta)}$

$\because\ f^{-1}(\alpha) = 5^{\alpha/5}$

$f^{-1}(\beta) = 5^{\beta/5}$

17. $f(n)=\frac{n^2(n^2+1)}{2} \quad g(n)=\left(\frac{n(n+1)}{2}\right)^2$

$\lim_{x\to\infty}\frac{x^2(x^2+1)4}{2(x+1)^2x^2}=2.$

18. $f(x)=\lim_{n\to\infty}\frac{x^{\frac{1}{n}}-1}{\frac{1}{n}}$

$\Rightarrow \quad f(x) = \ell n\ x$

$\Rightarrow \quad f(xy) = f(x) + f(y)$

20. $f(x)=\begin{cases}-\frac{1}{x}, & x\le -1\\ ax^2+b, & -1<x<1\\ \frac{1}{x}, & x\ge 1\end{cases}$

$f'(1^+)=\lim_{h\to 0}\frac{\frac{1}{1+h}-1}{h}=\lim_{h\to 0}\frac{-h}{h(1+h)}=-1$

$f'(1^-)=\lim_{h\to 0}\frac{a(1+h)^2+b-1}{h}$

$=\lim_{n\to 0}\frac{a+b-1+2ah+h^2}{h}$

$\Rightarrow a+b=1$ {$\because$ function differentiable at $x = 1$}

$f'(1^-) = 2a$

$f'(1^-)=f'(1^+) \Rightarrow 2a=-1 \Rightarrow a=-\frac{1}{2}$

$a + b = 1 \Rightarrow b=\frac{3}{2}$.

22. $x \to 0^+ \Rightarrow \cot^{-1}\frac{1}{x} \to 0$

$\therefore \quad \lim_{x\to 0^+}\sin^{-1}\sin\cot^{-1}\left(\frac{1}{x}\right) = 0$

$x \to 0^- \Rightarrow \cot^{-1}\frac{1}{x} \to \pi$

$\therefore \quad \lim_{x\to 0^-}\sin^{-1}\sin\cot^{-1}\left(\frac{1}{x}\right) = 0.$

26. $\frac{dy}{dx} = 1 + e^x \Rightarrow \quad \frac{dx}{dy}=\frac{1}{1+e^x}$

$\frac{d^2x}{dy^2}=\frac{d}{dx}\left(\frac{1}{1+e^x}\right)\frac{dx}{dy}=\frac{-e^x}{(1+e^x)^2}.\frac{1}{(1+e^x)}$

$\therefore \quad \left.\frac{d^2x}{dy^2}\right|_{x=1}=-\frac{e}{(1+e)^3}$

27. $f(x)=\begin{cases}(x-1)\sin\left(\frac{1}{x-1}\right), & x\ne 1\\ 0, & x=1\end{cases}$

Checking derivability of $f(x)$ at $x = 1$

$f'(1^+)=\lim_{h\to 0}\frac{h\sin\left(\frac{1}{h}\right)-0}{h}$

$\Rightarrow$ f is non derivable at x = 1

The function is continuous & derivable at x = 0.

28. $f(g(x)) = x$

$f'(g(x)) \cdot g'(x) = 1$

$f'(g(m)) \cdot g'(m) = 1$

$f'(b) \cdot 4 = 1$

$g'(b) = \frac{1}{4}$

29. $f(x) = \sqrt{x^2+1}$, $f'(x) = \frac{x}{\sqrt{x^2+1}}$

$h'(x) = 2$

$h'(g'(x)) = 2$ {$\because$ h'(x) is constant function}

$f'(h'(g'(x))) = f'(2) = \frac{2}{\sqrt{5}}$

31. $I = \frac{1}{3}\int \frac{x^3(3x^2dx)}{\sqrt{1+x^3}}$

$1 + x^3 = t \Rightarrow 3x^2dx = dt$

$\therefore \quad I = \frac{1}{3}\int \frac{(t-1)}{\sqrt{t}}dt$

$3I = \int \sqrt{t}dt - \int (t)^{-1/2}dt = \frac{2t^{3/2}}{3} - 2t^{1/2}$

$\therefore \quad I = \frac{2}{9}(1+x^3)^{3/2} - \frac{2}{3}(1+x^3)^{1/2} + c$

33. $\frac{1}{2}\int 2x\sqrt{\frac{2\sin(x^2+1)-\sin 2(x^2+1)}{2\sin(x^2+1)+\sin 2(x^2+1)}}dx$

$x^2 + 1 = t \Rightarrow 2xdx = dt$

$\Rightarrow \quad \frac{1}{2}\int\sqrt{\frac{2\sin t - \sin 2t}{2\sin t + \sin 2t}}dt$

$= \frac{1}{2}\int\sqrt{\frac{2-2\cos t}{2+2\cos t}}dt$

$= \frac{1}{2}\int \tan\frac{t}{2}dt = \frac{\frac{1}{2}\ell n\left|\sec\frac{t}{2}\right| + c}{\frac{1}{2}}$

$\Rightarrow \quad \log\left|\sec\left(\frac{x^2+1}{2}\right)\right| + c$

35. Putting x = 1 + h

$\ell = \lim_{h\to 0} 2^{2^{-\frac{1}{h}}} = 2^\circ = 1$

$m = \lim_{h\to 0}\frac{(1+h)\sinh}{h} = 1$

$\therefore \quad \int_1^1 \frac{\ell n(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx = 0$

$\because \quad \int_a^a f(x)\,dx = 0$

36. $\int_{-T/2}^{3T/2} f(x)dx = 18 \Rightarrow \int_0^{2T} f(x)dx = 18$

$\Rightarrow \int_0^T f(x)dx = 9$

$\int_{-a}^{a+5T} f(x)dx = \int_{-a}^{a} f(x)dx + \int_a^{a+5T} f(x)dx$

$= 2\int_0^a f(x)dx + 5\int_0^T f(x)dx = 2\times 3 + 5\times 9 = 51$

37. $\int \frac{x^3}{\sqrt{1+2x^2}}dx$

Putting $1 + 2x^2 = t^2 \quad \Rightarrow \quad 4xdx = 2tdt$

$\int \frac{\frac{t^2-1}{2}\cdot\frac{t}{2}dt}{t} \Rightarrow \frac{1}{4}\int(t^2-1)dt$

$\Rightarrow \frac{1}{4}\left(\frac{t^3}{3} - t\right) + c \Rightarrow \frac{1}{12}t\left(t^2-3\right) + c$

$\Rightarrow \frac{1}{6}(1+2x^2)^{1/2}(x^2-1) + c$

$f(x) = \frac{1}{6}(1+2x^2)^{1/2}(x^2-1) + c$

which passes through (1, 2)

$\therefore \quad 2 = 0 + c \qquad \therefore \quad c = 2$

$f(x) = \frac{1}{6}(1+2x^2)^{1/2}(x^2-1) + 2$

$\therefore \quad m + n = 6 + 2 = 8$

38. $|f(x) + 6 - x^2| = |f(x)| + |4 - x^2| + 2$

$|f(x) + 4 - x^2 + 2| = |f(x)| + |4 - x^2| + 2$

$f(x).(4 - x^2) \,.\, 2 \geq 0$

$\because \; f(x) \geq 0 \;\therefore\; 4 - x^2 \geq 0 \Rightarrow x^2 - 4 \geq 0$

$x \in [-2, 2]$

39. $I = \int_0^4 f(\gamma)d\gamma + \int_4^8 f(\gamma)d\gamma$

$\gamma = 8 - t \Rightarrow d\gamma = -dt$

$\therefore \; I = \int_0^4 f(\gamma)d\gamma + \int_4^0 f(8-t)(-dt)$

$= \int_0^4 f(\gamma)d\gamma + \int_0^4 f(\gamma)d\gamma = 16$

40. $\int_0^a f(x)dx = \tan^{-1} a$

$f(a) = \dfrac{1}{1+a^2} \Rightarrow \dfrac{1}{1+x^2} - 1 = \tan^2 x$

$\Rightarrow \sec^2 x = \dfrac{1}{1+x^2 \leq 1} \Rightarrow x = 0$

41. $f(x) = \int\left(\tan^2(\ell nx) + \tan(\ell nx) + \dfrac{1}{4}\right)dx$

$= \int\left(1 + \tan^2(\ell nx) + \tan(\ell nx) - \dfrac{3}{4}\right)dx$

$= \int\left(\tan(\ell nx) + x\dfrac{\sec^2(\ell nx)}{x}\right)dx - \int \dfrac{3}{4}dx$

$f(x) = x\tan(\ell nx) - \dfrac{3x}{4} + C$

$f(1) = -\dfrac{3}{4} + C = 0 \Rightarrow C = \dfrac{3}{4}$

$f(x) = x\tan(\ell nx) - \dfrac{3x}{4} + \dfrac{3}{4}$

$f\left(e^{\pi/4}\right) = e^{\pi/4} - \dfrac{3e^{\pi/4}}{4} + \dfrac{3}{4} = \dfrac{e^{\pi/4}+3}{4}$

42. $\dfrac{dx}{dt} = e^t\left\{\dfrac{2\tan t + 1 - \tan^2 t}{\sec^2 t}\right\}$

$\dfrac{dx}{dt} = e^t\{\sin 2t + \cos 2t\}$

$\dfrac{dy}{dt} = e^t\left\{\dfrac{1 - \tan^2 t - 2\tan t}{(1+\tan^2 t)}\right\}$

$= e^t\{\cos 2t - \sin 2t\}$

$\dfrac{dy}{dx} = \dfrac{\cos 2t - \sin 2t}{\cos 2t + \sin 2t} \Rightarrow \left.\dfrac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -1$

44. $L = \lim\limits_{x\to 0} e^{\tan x \ell n|x|} = \lim\limits_{x\to 0} e^{\frac{\ell n|x|}{\cot x}} = e^{\lim\limits_{x\to 0}\frac{\ell n|x|}{\cot x}}$

$= e^{\lim\limits_{x\to 0}\frac{-1}{x.\text{cosec}^2 x}}$ {by L-Hôpital Rule}

$= e^0 = 1$

45. Put $\sqrt[3]{\dfrac{2-x}{2+x}} = t \Rightarrow \dfrac{2-x}{2+x} = t^3$

$x = \dfrac{2-2t^3}{1+t^3} \Rightarrow dx = \dfrac{-12t^2}{(1+t^3)^2}dt$

$I = -\int \dfrac{2(1+t^3)^2 12t^3}{16t^6(1+t^3)^2}dt = -\dfrac{3}{2}\int\dfrac{dt}{t^3} = \dfrac{3}{4t^2} + c$

converting in x

$I = \dfrac{3}{4}\sqrt[3]{\left(\dfrac{2+x}{2-x}\right)^2} + c$

$f(x) = \dfrac{2+x}{2-x} = \dfrac{4-(2-x)}{2-x} = \dfrac{4}{2-x} - 1$

$f'(x) = \dfrac{4}{(2-x)^2} \Rightarrow f'(1) = 4$

48. $L = \lim\limits_{x\to 0^+} e^{\ell n(\text{cosec } x)^{1/\ell n\, x}}$

$= \lim\limits_{x\to 0^+} e^{\frac{\ell n(\text{cosec } x)}{\ell n\, x}} = \lim\limits_{x\to 0^+} e^{\frac{-\cot x}{1/x}} = \lim\limits_{x\to 0^+} e^{\frac{-x}{\tan x}} = e^{-1}$

$\Rightarrow \ell nL = -1$

49. $P = \lim\limits_{n\to\infty}\dfrac{\left[(1+n)(2+n)...(n+n)\right]^{1/n}}{n}$

$P = \lim\limits_{n\to\infty}\left\{\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{n}\right)....\left(1+\dfrac{n}{n}\right)\right\}^{1/n}$

$\Rightarrow \ell nP = \lim\limits_{n\to\infty}\dfrac{1}{n}\sum_{r=1}^{n}\ell n\left(1+\dfrac{r}{n}\right)$

$\Rightarrow \ell nP = \int_0^1 \ell n(1+x)dx$

$\Rightarrow \ \ell nP = \left(x\ell n(1+x)\right)_0^1 - \int_0^1 \frac{x+1-1}{x+1}dx$

$\Rightarrow \ \ell nP = \ell n2 - \int_0^1 dx + \int_0^1 \frac{dx}{x+1}$

$\Rightarrow \ \ell nP = \ell n2 - 1 + \left[\ell n(x+1)\right]_0^1$

$\Rightarrow \ \ell nP = \ell n\left(\frac{4}{e}\right) \ \Rightarrow \ P = \frac{4}{e}$

$\Rightarrow \ \log_2(eP) = \log_2(4) = 2$

52. Range of $\text{cosec}^{-1}x = \{-2, -1, 0, 1\}$

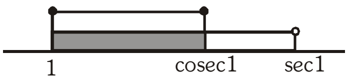
Range of $\sec^{-1}x = \{0, 1, 2, 3\}$

for the given inequality to satisfy

$[\text{cosec}^{-1}x] = 1$ & $[\sec^{-1}x] = 0$

$1 \le \text{cosec}^{-1}x < 2$ & $0 \le \sec^{-1}x < 1$

$1 \le x \le \text{cosec}1$ & $1 \le x < \sec 1$



53. $y = \lim_{x\to\frac{\pi}{2}} (\cot x)^{\cos x}$

$\ell ny = \lim_{x\to\frac{\pi}{2}} \cos x \ \ell n(\cot x) = \lim_{x\to\frac{\pi}{2}} \frac{\ell n\cot x}{\sec x}$

$= \lim_{x\to\frac{\pi}{2}} \frac{(-\text{cosec}^2x)}{(\cot x)\sec x\tan x}$

$\ell ny = 0 \Rightarrow y = e^0 = 1$

$y = \lim_{x\to\frac{\pi}{2}} (\cos x)^{\cot x} \Rightarrow \ell ny = \lim_{x\to\frac{\pi}{2}} \cot x \ \ell n\cos x$

$\ell ny = \lim_{x\to\frac{\pi}{2}} \frac{\ell n\cos x}{\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{(-\sin x)}{\cos x \ \sec^2 x}$

$= \frac{-\sin x.\cos^2 x}{\cos x}$

$\ell ny = \lim_{x\to\frac{\pi}{2}} \sin x \cos x = 0$

$y = e^0 = 1$

$= \lim_{x\to\frac{\pi}{2}} f(x) = 1$

If function is continuous at $x = \frac{\pi}{2}$, then

$\lim_{x\to\frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right) \ \Rightarrow \ f\left(\frac{\pi}{2}\right) = 1$

54. $f(x) = \sqrt{(x-1)+4-4\sqrt{x-1}} + \sqrt{(x-1)+9-6\sqrt{x-1}}$

$= \left|\sqrt{x-1}-2\right| + \left|\sqrt{x-1}-3\right|$

at $x=1.5$

$f(x) = \left(2-\sqrt{x-1}\right) + \left(3-\sqrt{x-1}\right) = 5 - 2\sqrt{(x-1)}$

$f'(x) = \frac{-2}{2\sqrt{x-1}}$

$f'(1.5) = -\frac{1}{\sqrt{\frac{3}{2}-1}} = -\sqrt{2}$

56. $\int x^x\left((\ell nx)^2 + \ell nx\right)dx + \int \underset{I}{x^x}\underset{II}{\left(\frac{1}{x}\right)}dx$ (By parts)

$= \int x^x\left((\ell nx)^2 + \ell nx\right)dx + x^x \ \ell nx - \int x^x(1+\ell nx)\ell nx dx$

$= x^x \ \ell n x + c$

57. $f(x) = \int \frac{\ell nx + \ell n^2x + 1}{(1+\ell nx)^2}dx$

$= \int \left(\frac{\ell nx}{1+\ell nx} + \frac{1}{(1+\ell nx)^2}\right)dx$

$= \int \left\{\frac{\ell nx}{1+\ell nx} + x\left(\frac{1}{x(1+\ell nx)^2}\right)\right\}dx = \frac{x\ell nx}{1+\ell nx} + C$

$f(x) = \frac{x\ell nx}{1+\ell nx} \ \{\because f(1) = 0\}$

$\Rightarrow x > 0 \ \& \ \ell nx \ne -1$

$\Rightarrow$ Domain of 'f' is $\left(0, \frac{1}{e}\right) \cup \left(\frac{1}{e}, \infty\right)$

58. Differentiating the given integral

$f(x)\cos x = f(x)f'(x)$

$f'(x) = \cos x \quad \Rightarrow \quad f(x) = \sin x + C$

$\Rightarrow f(x) = \sin x - 1 \ \left\{\because f\left(\frac{\pi}{2}\right) = 0\right\}$

$g(x) = \frac{\sin x - 1}{\sin x - 1}$

Period of g(x) is 2π as discontinuily repeats after interval of 2π

59. LHL : $\lim_{x\to 0^+} \frac{k^x + k^{-x} - 2}{x^2}$

$= \lim_{x\to 0^+} \frac{\left(k^x - k^{-x}\right)\ell nk}{2x}$ {using L'Hopital}

$\lim\limits_{x\to 0^+}\left(\dfrac{k^x+k^{-x}}{2}\right)\ell n^2 k = \ell n^2 k$

RHL : $\lim\limits_{x\to 0^-} 3\ell n(k-x)-2 = 3\ell nk - 2$

$\ell n^2 k = 3\ell nk - 2 \Rightarrow \ell n^2 k - 3\ell nk + 2 = 0$

$(\ell nk-1)(\ell nk-2)=0 \Rightarrow k = e \text{ or } e^2$

60. $AB = \begin{bmatrix} 4x^2 & 6 & 0 \\ 1 & -5 & 1 \\ 2 & 0 & 8x \end{bmatrix}$

tr(AB) = $4x^2 + 8x - 5 = f(x)$

$\therefore\ 3\int \dfrac{dx}{4x^2+8x-5}$

$=\dfrac{3}{4}\int\dfrac{dx}{x^2+2x+1-\dfrac{9}{4}} = \dfrac{3}{4}\int\dfrac{dx}{(x+1)^2-(3/2)^2}$

$=\dfrac{3}{4}\cdot\dfrac{2}{2.3}\ell n\left|\dfrac{x+1-3/2}{x+1+3/2}\right|+c = \dfrac{1}{4}\ell n\left|\dfrac{2x-1}{2x+5}\right|+c$

61. $F(x)=\int e^{\sin^{-1}x}\left(1-\dfrac{x}{\sqrt{1-x^2}}\right)dx$

$\sin^{-1}x = t \Rightarrow x = \sin t \Rightarrow dx = \cos t\,dt$

$\therefore\ F(x)=\int e^t\left(1-\dfrac{\sin t}{\cos t}\right).\cos t\,dt$

$=\int e^t(\cos t-\sin t)\ dt = e^t\cos t + c$

$F(x)=e^{\sin^{-1}x}\cos(\sin^{-1}x)+c$

$F(0) = 1 + c = 1 \Rightarrow c = 0$

$\therefore\ F(x)=e^{\sin^{-1}x}\cos(\sin^{-1}x)$

$F\left(\dfrac{1}{2}\right)=e^{\pi/6}\cos\dfrac{\pi}{6}=\dfrac{e^{\pi/6}.\sqrt{3}}{2}=\dfrac{k.\sqrt{3}e^{\pi/6}}{\pi} \Rightarrow k=\dfrac{\pi}{2}$

62. $I=\int\dfrac{\sqrt{9-x^2}}{x^4}dx=\dfrac{1}{-18}\int 9\sqrt{(9x^{-2}-1)}(-2x^{-3})dx$

$9x^{-2}-1=t \Rightarrow -18x^{-3}dx = dt$

$I=-\dfrac{1}{18}\int t^{1/2}dt = -\dfrac{1}{18}\cdot\dfrac{t^{3/2}.2}{3}+c$

$=-\dfrac{1}{27}(9x^{-2}-1)^{3/2}+c=-\dfrac{(9-x^2)^{3/2}}{27x^3}+c$

$\dfrac{k}{9}=-\dfrac{1}{27} \Rightarrow k=-\dfrac{1}{3}$

64. $f\left(\dfrac{2}{x-2}\right)=\dfrac{1}{\dfrac{4}{(x-2)^2}-\dfrac{34}{(x-2)}+66}$

$=\dfrac{(x-2)^2}{4-34(x-2)+66(x-2)^2}=\dfrac{(x-2)^2}{2(3x-7)(11x-24)}$

$\Rightarrow f\left(\dfrac{2}{x-2}\right)$ is discontinuous at $x = 2, \dfrac{7}{3}, \dfrac{24}{11}$

68. $g(x) = f(e^x) . e^{f(x)}$

$g'(x) = f'(e^x) . e^x . e^{f(x)} + f(e^x) . e^{f(x)} . f'(x)$

$g'(0) = f'(1) . e^{f(0)} + f(1) . e^{f(0)} . f'(0) = 2$

69. Put $\log\{f(x) . g(x)\} = t$

70. Put $1 + x^{4/5} = t$

71. $I=\int_0^2\sqrt{\dfrac{2+x}{2-x}}$ Put $x = 2\cos\theta \Rightarrow dx = -2\sin\theta\, d\theta$

at $x = 0,\ \theta = \pi/2$ & $x = 2,\ \theta = 0$

$I = 4\int_0^{\pi/2}\dfrac{\cos\theta/2}{\sin\theta/2}\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}d\theta$

$=2\int_0^{\pi/2}(1+\cos\theta)=\pi+2$

73. $\int\dfrac{\text{cosec}^2x-2009}{\cos^{2009}x}dx$

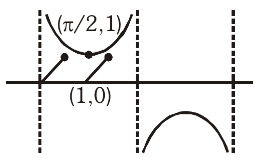
$=\int\text{cosec}^2x.(\cos x)^{-2009}dx-2009\int\dfrac{1}{(\cos x)^{2009}}dx$

$= I_1 - I_2$

Applying by parts on I_1,

we get $\int\dfrac{\text{cosec}^2x-2009}{\cos^{2009}x}dx$

$=-\dfrac{\cot x}{(\cos x)^{2009}}+c$



$\therefore$ A(x) = cotx and B(x) = cosx

$\Rightarrow \dfrac{A(x)}{B(x)}=\text{cosec}x=\{x\}$ for $x\in[0,2\pi]$ the equation has no solution as clear from the graph

74. $g'(x)=\ell n(\sec x\tan x-\sec^2x+1)\ge 0$

$\Rightarrow$ secx tanx – $\sec^2x + 1 \ge 1$

$\Rightarrow \dfrac{\sin x-1}{\cos^2x}\ge 0 \Rightarrow \sin x - 1 \ge 0$

Hence there is no value of x in $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ which satisfies above inequality.

77. $x^2 + y^2 = 5$

$2x + 2yy' = 0$

$\Rightarrow \quad y' = -\dfrac{x}{y} \qquad \text{....... (i)}$

$\Rightarrow \quad y'' = -\left(\dfrac{y.1 - x.y'}{y^2}\right) = -\left(\dfrac{x^2+y^2}{y^3}\right) \quad \{\text{from (i)}\}$

$\Rightarrow \quad y'' = -\dfrac{5}{y^3}$

$\Rightarrow \quad y'' + 5y^{-3} = 0 \Rightarrow \quad 2y'' + 10y^{-3} = 0$

$\therefore \quad k = 10$

78. $I = \int_{-1}^{0}\left[\dfrac{-1}{1+x^2}\right]dx + \int_{0}^{1} 0.dx + \int_{1}^{2}\left[\dfrac{1}{1+x^2}\right]dx$

$I = -\int_{-1}^{0} dx = -1$

80. $I = \int \dfrac{dx}{\sqrt[3]{(\cos^4 x)^3 + 3\cos^4 x.\cos^2 x(\cos^4 x + \cos^2 x) + (\cos^2 x)^3}}$

$I = \int \dfrac{dx}{\cos^4 x + \cos^2 x} = \int \dfrac{dx}{(\cos^2 x)(\cos^2 x + 1)}$

$= \int \left(\dfrac{1}{\cos^2 x} - \dfrac{1}{(\cos^2 x + 1)}\right)dx$

$= \tan x - \int \dfrac{\sec^2 x}{\tan^2 x + 2}dx$

$= \tan x - \dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{\tan x}{\sqrt{2}}\right) + c$

$f(x) = \tan x$

82. Let $\dfrac{2x+3}{2x-3} = t \Rightarrow \quad x = \dfrac{3}{2}\left(\dfrac{1+t}{t-1}\right)$

$\therefore \quad f(t) = \dfrac{2}{t-1} \Rightarrow \quad \int f(x)dx = \ell n(x-1)^2 + c$

84. $\int_0^{\pi} \underset{I}{f(x)} \underset{II}{\cos x}\, dx + \int_0^{\pi} f''(x)\cos x dx = 3$

$\Rightarrow \quad [f(x)\sin x]_0^{\pi} - \int_0^{\pi} \underset{I}{f'(x)} \underset{II}{\sin x}\, dx + \int_0^{\pi} f''(x)\cos x\, dx = 3$

$\Rightarrow [f'(x)\cos x]_0^{\pi} - \int_0^{\pi} f''(x)\cos x dx + \int_0^{\pi} f''(x)\cos x\, dx = 3$

$\Rightarrow \quad -f'(\pi) - f'(0) = 3 \qquad \Rightarrow \quad f'(\pi) = -1$

85. $e^{f(x)} . f'(x) = 2x$

$e^{f(x)} = x^2 + c$

{Integrating both sides with respect to x}

$f(x) = \ell n(x^2 + c)$

$f'(x) = \dfrac{2x}{x^2+c}$

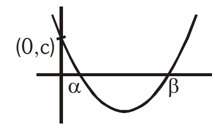
$f'(1) = \dfrac{2}{1+c} = 1 \qquad \{\because \; f'(1) = 1\}$

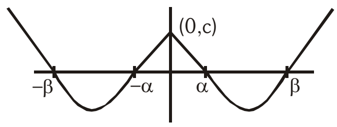
$\therefore \quad c = 1$

$\therefore \quad f(x) = \ell n(x^2 + 1)$

$f'(x) = \dfrac{2x}{x^2+1}$

87. $y = f(x)$ $\qquad y = f(|x|)$

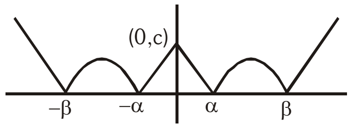




$\therefore \quad I_1 = \int_{-\beta}^{\beta} f(|x|)dx = 2\int_0^{\beta} f(x)dx$

$= 2\int_0^{\alpha} f(x)dx + 2\int_{\alpha}^{\beta}(f(x))dx$

$y = |f(|x|)|$



$I_2 = \int_{-\beta}^{\beta} |f(|x|)|dx = 2\int_0^{\beta}|f(x)|dx$

$= 2\int_0^{\alpha} f(x)dx + 2\int_{\alpha}^{\beta}(-f(x))dx$

$\Rightarrow \quad I_1 + I_2 = 4\int_0^{\alpha} f(x)dx$

88. $f^2(x) + xf(x) = 3,$

Differentiating w.r.t. x,

$\Rightarrow \quad 2f(x)f'(x) + f(x) + xf'(x) = 0$

$\Rightarrow \quad f'(x) = -\dfrac{f(x)}{2f(x)+x} \qquad \text{......(i)}$

Now $\quad I = \int \dfrac{3x^3 + 6x^2 f(x) + 2f(x)}{(2f(x)+x)(x^3 - 2f(x))^2}dx$

$= \int \dfrac{\left(3x^2 + \dfrac{2f(x)}{2f(x)+x}\right)(2f(x)+x)}{(2f(x)+x)(x^3-2f(x))^2}dx$

$= \int \dfrac{3x^2 + \dfrac{2f(x)}{2f(x)+x}}{(x^3 - 2f(x))^2}dx$

From (i), $I = \int \dfrac{3x^2 - 2f'(x)}{(x^3 - 2f(x))^2}dx = \dfrac{-1}{x^3 - 2f(x)} + c$

89. $y=\frac{x-1}{x+2} \Rightarrow x=\frac{2y+1}{1-y}$

$\therefore \quad f^{-1}(x)=\frac{2x+1}{1-x}$

$\Rightarrow \quad \frac{d}{dx}(f^{-1}(x))=\frac{(1-x)2+(2x+1)}{(1-x)^2}=\frac{3}{(1-x)^2}$

90. $I=\int_{-4}^{4}\cot^{-1}x\,dx=\int_{-4}^{4}\cot^{-1}(-x)dx=\int_{-4}^{4}(\pi-\cot^{-1}x)dx$

or $I=\int_{-4}^{4}\pi dx-\int_{-4}^{4}\cot^{-1}x\,dx$ or $2I=8\pi$

$\therefore \quad I=4\pi$

92. $\int_2^x f(t)dt=\frac{x^2}{2}+\int_x^2 t^2 f(t)dt$

Differentiating w.r.t. x, we get

$f(x)(1+x^2)=x \Rightarrow f(x)=\frac{x}{1+x^2}$, which is an odd function.

Now $\int_{-\pi/4}^{\pi/4}\frac{f(x)+x^9-x^3+x+1}{\cos^2 x}dx$

$=\int_{-\pi/4}^{\pi/4}\frac{\frac{x}{1+x^2}+x^9-x^3+x}{\cos^2 x}dx+\int_{-\pi/4}^{\pi/4}\sec^2 x dx$

$=0+2$

93. Clearly $f(x)=x$

Let $g(x)=x^2+\int_0^x(\sin t+a^2t^3+bt)dt-\alpha$

As $|x|\to\infty, \quad g(x)\to\infty$

At $x=0, \quad g(x)=-\alpha<0$

Hence atleast two points of intersection will be obtained.

94. $I_n=\int_1^{e^2}(\log_e x)^n d(x^2)=\int_1^e 2x(\log_e x)^n dx$

$=\left(2(\log_e x)^n\frac{x^2}{2}\right)_1^e-2\int_1^e\frac{n(\log_e x)^{n-1}}{x}\cdot\frac{x^2}{2}dx$

$=e^2-\frac{n}{2}I_{n-1}$

$I_n+\frac{n}{2}I_{n-1}=e^2$

95. $\frac{x^3}{3}-4x^2+13x=x\sin\frac{a}{x}$

$\Rightarrow \quad \sin\frac{a}{x}=\frac{1}{3}(x^2-12x+39)$

as $\frac{1}{3}[(x-6)^2+3]\ge 1 \Rightarrow \sin\frac{a}{x}=1$

so for solution $x=6$ & $\frac{a}{6}=(4K+1)\frac{\pi}{2}$

$\Rightarrow \quad a=3(4K+1)\pi.$

100. $\tan(\sec^{-1}x)=\sin\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$

from the given equation it is clear that x is positive.

Let $\sec^{-1}x=\theta$

$\Rightarrow \quad \sec\theta=x \qquad \Rightarrow \quad \tan\theta=\frac{2}{\sqrt{5}}$

$\Rightarrow \quad x^2=1+\frac{4}{5} \qquad \Rightarrow \quad x=\frac{3}{\sqrt{5}}.$

104. Option (A) : $f(x) \to$ odd degree

$\Rightarrow f'(x) \to$ even degree

Even degree polynomial does not necessarily have real roots

Option (B) : $P(x)=(x-2)^3 Q(x)$

$P'(x)=(x-2)^3Q'(x)+3(x-2)^2Q(x)$

$=(x-2)^2\{(x-2)Q'(x)+3Q(x)\}$

Option (B) is corret

Option (C) : If $f(x)$ is a differentiable function which is even, then f'(x) is an odd function.

Option (C) is correct

Option (D) :

$y=\sin^{-1}\left\{\cos\left(\frac{\pi}{2}-\cos^{-1}x\right)\right\}+\cos^{-1}\left(\sin\left(\cos^{-1}x\right)\right)$

$=\sin^{-1}\{\sin(\cos^{-1}x)\}+\cos^{-1}\{\sin(\cos^{-1}x)\}$

$y=\pi/2$

option (D) is correct

105. $I_n=\int_0^1(1-x^2)^n dx$

$=(1-x^2)^n\cdot x\Big|_0^1+2n\int_0^1(1-x^2)^{n-1}x^2dx$

$=2n\int_0^1(1-x^2)^{n-1}\{1-(1-x^2)\}dx$

$I_n = 2nI_{n-1} - 2nI_n \Rightarrow \quad I_n = \dfrac{2n}{2n+1} I_{n-1}$

$I_1 = \int_0^1 (1-x^2)dx = \dfrac{2}{3}$

$I_2 = \dfrac{4}{5}.\dfrac{2}{3}, \quad I_3 = \dfrac{6.4.2}{7.5.3}$

$\vdots$

$I_n = \dfrac{2.4.6.....2n}{3.5.7.......(2n+1)} = \dfrac{2^n n!}{3.5.7.......(2n+1)}$

106. (A) $\tan^{-1}\left(\tan\left(\dfrac{47\pi}{7} - 7\pi\right)\right) = -\dfrac{2\pi}{7}$

(B) $1 + \sin x \in [0,2\,] \subset [0, \pi], \forall\, x \in R$
$\Rightarrow \cos^{-1}(\cos(1+\sin x)) = 1 + \sin x$

(C) $\sin^{-1}\left(\sin\dfrac{23\pi}{5}\right) = \sin^{-1}\left(\sin\left(5\pi - \dfrac{23\pi}{5}\right)\right) = \dfrac{2\pi}{5}$

$\cos^{-1}\left(\cos\dfrac{23\pi}{5}\right) + \cos^{-1}\left(\cos\left(\dfrac{23\pi}{5} - 4\pi\right)\right) = \dfrac{3\pi}{5}$

(D) $\tan^{-1}2 > \dfrac{\pi}{2} - \tan^{-2} \Leftrightarrow \tan^{-1}2 > \dfrac{\pi}{4}$
$\Leftrightarrow \tan^{-1}2 > \tan^{-1}1$ which is true

108. Let $\cos^{-1}x = \theta \in [0, \pi]$

$\Rightarrow \cos^{-1}\sqrt{\dfrac{1+x}{2}}$

$= \cos^{-1}\left(\cos\dfrac{\theta}{2}\right) = \dfrac{\theta}{2} = \dfrac{1}{2}\cos^{-1}x$

also $\sin^{-1}\sqrt{\dfrac{1-x}{2}} = \sin^{-1}\left(\sin\dfrac{\theta}{2}\right) = \dfrac{1}{2}\cos^{-1}x$

$\therefore$ A, C are correct.

112. For $x > 2$

$f(x) = \int_0^1 (6-t)dt + \int_1^x (t+4)dt$

$= \left(6t - \dfrac{t^2}{2}\right)_0^1 + \left(\dfrac{t^2}{2} + 4t\right)_1^x$

$= \left(6 - \dfrac{1}{2}\right) + \left(\dfrac{x^2}{2} + 4x\right) - \left(\dfrac{1}{2} + 4\right)$

$= \dfrac{11}{2} + \dfrac{x^2}{2} + 4x - \dfrac{9}{2} = \dfrac{x^2}{2} + 4x - 1$

$f(x) = \begin{cases} 5x+1, & x \le 2 \\ \dfrac{x^2}{2} + 4x - 1, & x > 2 \end{cases}$

$f'(x) = \begin{cases} 5, & x < 2 \\ x+4, & x > 2 \end{cases}$

$f'(2^-) \ne f'(2^+) \quad \Rightarrow f$ is not differentiable

113. Let $I_1 = \int_{1/e}^{\tan x} \dfrac{t}{1+t^2}dt$

Put $t = \dfrac{1}{u} \Rightarrow I_1 = \int_e^{\cot x} \dfrac{-du}{u(u^2+1)}$

$= \int_{\cot x}^{e} \dfrac{dt}{t(t^2+1)}$

$\therefore \int_{1/e}^{\tan x} \dfrac{t}{1+t^2}dt + \int_{1/e}^{\cot x} \dfrac{dt}{t(1+t^2)} = \int_{1/e}^{e} \dfrac{dt}{t(1+t^2)}$

$= \int_{1/e}^{e} \left(\dfrac{1}{t} - \dfrac{t}{t^2+1}\right)dt = 1$

(C) $\lim\limits_{x\to 0} \dfrac{\int_0^{x^2} \cos t^2 dt}{x \sin x} = \lim\limits_{x\to 0} \dfrac{\cos x^4 .2x}{x\cos x + \sin x} = 1$

(D) $I = \lim\limits_{n\to\infty} \left\{\tan\dfrac{\pi}{2n}........\tan\dfrac{n\pi}{2n}\right\}^{1/n}$

$\Rightarrow \ell nI = \lim\limits_{n\to\infty} \dfrac{1}{n} \ell n\left\{\tan\dfrac{\pi}{2n}\tan\dfrac{2\pi}{2n}........\tan\dfrac{n\pi}{2n}\right\}$

$= \lim\limits_{n\to\infty} \dfrac{1}{n}\sum_{r=1}^{n} \ell n \tan\dfrac{r\pi}{2n} = \int_0^1 \ell n \tan\dfrac{\pi x}{2}dx$

$= \dfrac{2}{\pi}\int_0^{\pi/2} \ell n \tan\theta d\theta = 0$

$\therefore \quad I = 1$

114. Statement 1 : as y is onto

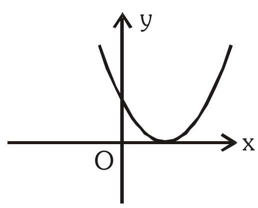
$\therefore \quad x^2 + x + a \in [0, \infty)$

so min. value of $x^2 + x + a = 0$

$\Rightarrow \quad -\dfrac{D}{4A} = 0$

$\Rightarrow \quad a - \dfrac{1}{4} = 0$

$\Rightarrow a = \dfrac{1}{4}$



Statement 2 : $y = \left(x + \dfrac{1}{2}\right)^2 + \left(a - \dfrac{1}{4}\right)$

so $y_{min} = a - \dfrac{1}{4}$

116. $\alpha_1\,\alpha_2\,\alpha_3 = -\dfrac{d}{a}$

Case-I : when $\alpha_1, \alpha_2, \alpha_3 < 0$

then $\dfrac{\pi}{2} < \cos^{-1}\alpha_1 < \pi;\ \dfrac{\pi}{2} < \cos^{-1}\alpha_2 < \pi;$

$\dfrac{\pi}{2} < \cos^{-1}\alpha_3 < \pi$

$\dfrac{3\pi}{2} < \cos^{-1}\alpha_1 < \cos^{-1}\alpha_2 + \cos^{-1}\alpha_3 < 3\pi$

but $\cos^{-1}\alpha_1 + \cos^{-1}\alpha_2 + \cos^{-1}\alpha_3 = \pi$ which is not possible

therefore all the three roots will not be negative

Case-II: $\alpha_1, \alpha_2 > 0\ \ \&\ \ \alpha_3 < 0$

$0 < \cos^{-1}\alpha_1 < \dfrac{\pi}{2},\ \ 0 < \cos^{-1}\alpha_2 < \dfrac{\pi}{2},\ \ \dfrac{\pi}{2} < \cos^{-1}\alpha_3 < \pi$

$\dfrac{\pi}{2} < \cos^{-1}\alpha_1 + \cos^{-1}\alpha_2 + \cos^{-1}\alpha_3 < 2\pi$

$\therefore$ In this case

$\cos^{-1}\alpha_1 + \cos^{-1}\alpha_2 + \cos^{-1}\alpha_3 = \pi$

is possible.

118. Statement 1 : $f(x) = \dfrac{1}{x}$ its inverse is $f^{-1}(x) = \dfrac{1}{x}$

$f(x) = f^{-1}(x) \Rightarrow x \in R_0 \not\Rightarrow f(x) = x$

as $f(x) = x$ holds only on $x = \pm 1$

Statement 2 : $f^{-1}(x) = x \Rightarrow f(f(x)) = f(x)$

$x = f(x)$

121. $f(x)\begin{cases} (x-1)^2(x-3)-(x-2)^3+\tan x, & x<1 \\ -(x-1)^2(x-3)-(x-2)^3+\tan x, & 1 \le x < 2 \\ -(x-1)^2(x-3)+(x-2)^3+\tan x, & 2 < x < 3 \\ (x-1)^2(x-3)+(x-2)^3+\tan x, & x \ge 3 \end{cases}$

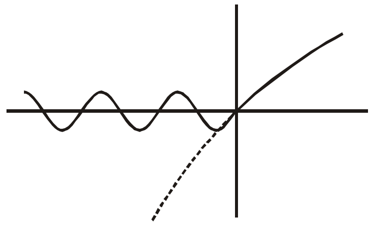
$f(x)\begin{cases} 2(x-1)(x-3)+(x-1)^2-3(x-2)^2+\sec^2 x, & x<1 \\ -2(x-1)(x-3)-(x-1)^2-3(x-2)^2+\sec^2 x & 1<x<2 \\ -2(x-1)(x-3)-(x-1)^2+3(x-2)^2+\sec^2 x, & 2<x\le 3 \\ (x-1)^2+2(x-1)(x-3)+3(x-2)^2+\sec^2 x, & x \ge 3 \end{cases}$

Clearly at x = 3 function is non-derivable.

zero at $x = \dfrac{\pi}{2}\ \&\ \dfrac{3\pi}{2}$ function is discontinuous hence non-derivable.

Also statement-2 is true & explains statement-1

122. Statement-1



Consider $f(x) = \begin{cases} \sin x & x<0 \\ \ell n(1+x), & x \ge 0 \end{cases}$

Although the function is not periodic

but still $f(x) = 0$ has infinite number of solutions

$\Rightarrow$ statement 1 is false

Statement-2 is true.

123. Statement-1 :

Put $x = \dfrac{1}{t} \quad \Rightarrow \quad dx = -\dfrac{1}{t^2}dt$

$I = -\displaystyle\int_3^{1/3} t\,\text{cosec}^{99}\left(\dfrac{1}{t}-t\right)\dfrac{1}{t^2}dt$

$= -\displaystyle\int_{1/3}^{3} \dfrac{1}{t}\text{cosec}^{99}\left(t-\dfrac{1}{t}\right)dt$

$I = -I \quad \Rightarrow \quad 2I = 0 \ \Rightarrow\ I = 0$

124. $f(2-x) = f(2+x)$

$x \to x+2$

$f(-x) = f(x+4)$ .... (i)

$f(20-x) = f(x)$

$x \to x+4$

$f(16-x) = f(x+4)$ .... (ii)

From (i) & (ii)

$f(-x) = f(16-x)$

$f(x) = f(x+16)$ $\therefore$ Period of $f(x)$ is 16.

Statement-1 : $\displaystyle\int_4^{4+16} f(x)dx = \int_0^{16} f(x)dx = 10$

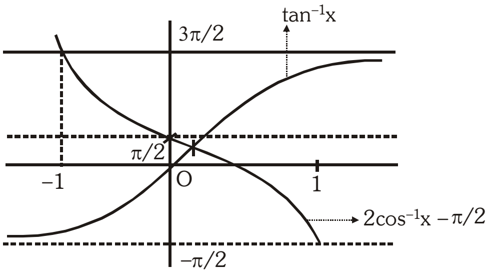
$\displaystyle\int_{-9}^{-9+160} f(x)dx = 10\int_0^{16} f(x)dx = 100$

$\therefore$ Statement-1 is false.

Paragraph for Question 132 to 134

132. $\cos^{-1}x - \sin^{-1}x = \tan^{-1}x$

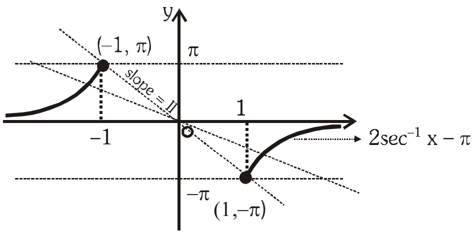
$2\cos^{-1}x - \dfrac{\pi}{2} = \tan^{-1}x$



Exactly one solution.

133. $\sec^{-1} x - \text{cosec}^{-1} x - \pi/2 = mx$

$2\sec^{-1}x - \pi = mx$



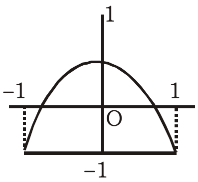
for exactly two solutions the slope (m) of the line must lie in [π, 0)

134. $y = \sin\left(\frac{\pi}{2} - 2\sin^{-1} x\right)$

$y = \cos(2\sin^{-1} x)$

$= 1 - 2\sin^2(\sin^{-1} x)$

$= 1 - 2x^2$.



Paragraph for Question 135 to 137

135. $\phi(x)$ is odd $\Rightarrow a = -1$

$\phi(x)$ is aperiodic $\Rightarrow b = 4$

$\phi(x)$ is one-one $\Rightarrow c = 5$

$\phi(x)$ is onto $\Rightarrow d = 7$

$k(\phi(x)) = 15$.

136. $h(x)$ is even $\Rightarrow a = 0$

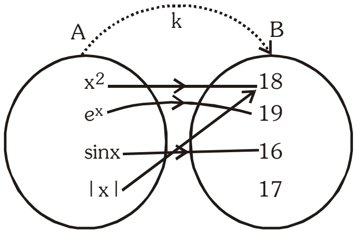
$h(x)$ is aperiodic $\Rightarrow b = 4$

$h(x)$ is many-one $\Rightarrow c = 6$

$h(x)$ is into $\Rightarrow c = 8$

$k(h(x)) = 18$.

137.



$k(x^2) = 0 + 4 + 6 + 8$

$= 18$

$k(e^x) = 2 + 4 + 5 + 8 \ \ 7$

$= 19$

$k(\sin x) = -1 + 3 + 6 + 8$

$= 16$

$k(|x|) = 0 + 4 + 6 + 8$

$= 18$.

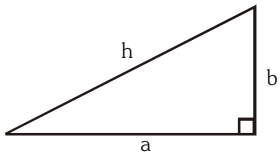
Paragraph for Question 138 to 140

$\frac{1}{2}ab = 25 \Rightarrow ab = 50$

$a + b + h = x$

$h = x - (a + b)$ ...(i)

Now, $a^2 + b^2 = h^2$



$(a + b)^2 - 2ab = h^2$

$(x - h)^2 - 100 = h^2$

$x^2 - 2xh + h^2 - 100 = h^2$

$h = \frac{x^2 - 100}{2x}$

138. $f(x) = \frac{x^2 - 100}{2x}$

$f(10) = f(-10) = 0$

$\Rightarrow$ f is many one function.

$y = \frac{x^2 - 100}{2x}$

$x^2 - 2y \cdot x - 100 = 0$

$D \geq 0$

$4y^2 + 400 \geq 0 \Rightarrow y \in R \Rightarrow$ onto.

139. $2 \cdot \log_2 3 \cdot \log_3 4 \ldots\ldots. \log_{31} 32$

$= \frac{2.\log 3}{\log 2} \frac{\log 4}{\log 3} \ldots\ldots. \frac{\log 32}{\log 31} = 2 \cdot \frac{\log 32}{\log 2}$

$= 2.5 = 10$

$\cos^{-1}\cos(f(10) + 10) = \cos^{-1}\cos 10$

$(\because f(10) = 0)$

$\Rightarrow [4\pi - 10] = 2$.

140. $f(x) = \frac{x^2 - 100}{2x}$

$\Rightarrow$ f is discontinuous at $x = 0$

$f(f(x)) = \frac{f^2(x) - 100}{2f(x)}$

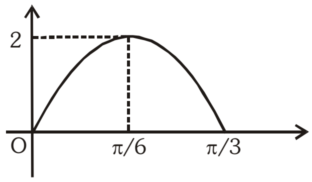
$\Rightarrow$ $f(f(x))$ is discontinuous at all x, where $f(x) = 0$

$\Rightarrow \quad \dfrac{x^2-100}{2x} = 0 \quad \Rightarrow \quad x = \pm 10.$

Sum of values of x where $f(f(x))$ is discontinuous is $0 + 10 - 10 = 0$.

Paragraph for Question 144 to 146

$g(x) = 2\sin 3x$



$$f(x) = \begin{cases} 2\sin 3x & 0 \le x \le \dfrac{\pi}{6} \\ 2 & \dfrac{\pi}{6} < x \le \dfrac{\pi}{3} \\ -x+2+\dfrac{\pi}{3} & \dfrac{\pi}{3} < x < 3 \\ \pi - x + 1 & 3 \le x \le 5 \end{cases}$$

144. $\lim\limits_{x \to \frac{\pi}{3}^-} f(x) = 2$

145. $\displaystyle\int_0^{\pi/3} f(x)dx = 2\int_0^{\pi/6} \sin 3x dx + \int_{\pi/6}^{\pi/3} 2.dx$

$$= -2\left(\frac{\cos 3x}{3}\right)_0^{\pi/6} + (2x)_{\pi/6}^{\pi/3}$$

$$= -\frac{2}{3}(0-1) + 2\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{2}{3} + \frac{\pi}{3} = \frac{\pi+2}{3}$$

146. $f(x) = \pi - x + 1,\ 3 \le x \le 5$

$f'(x) = -1$

$\therefore \quad f'(4) = -1$

Paragraph for Question 147 to 149

148. $\displaystyle\int \frac{\tan x dx}{(\tan^2 x + \tan x + 1)}$

$$= \int \frac{\tan x \sec^2 x dx}{(1+\tan^2 x)(1+\tan^2 x + \tan x)}$$

Let $\tan x = t$

$\sec^2 x dx = dt$

$$= \int \frac{t dt}{(1+t^2)(1+t^2+t)} = \int \frac{(1+t^2+t)-(1+t^2)}{(1+t^2)(1+t^2+t)} dt$$

$$= \int \frac{dt}{(t^2+1)} - \int \frac{dt}{(1+t^2+t)}$$

$$= \tan^{-1}(t) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)$$

$$= x - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x + 1}{\sqrt{3}}\right) + c$$

$$\Rightarrow \quad g(x) = \frac{2\tan x + 1}{\sqrt{3}}$$

$$= -\sqrt{3} < \tan x < \sqrt{3}$$

$$-\frac{2\sqrt{3}+1}{\sqrt{3}} < \frac{2\tan x + 1}{\sqrt{3}} < \frac{2\sqrt{3}+1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} - 2 < g(x) < \frac{1}{\sqrt{3}} + 2$$

149. $\displaystyle\int \frac{x^2 dx}{(x^4+x^2+1)} + \int x^2\left(\frac{1}{x^4} + \frac{1}{x^2} + 1\right) - \left(\frac{1}{x^2}\right)dx$

$$= \int \frac{(x^2-1)dx}{(x^4+x^2+1)} = \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x^2+\frac{1}{x^2}+1\right)}$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x^2+\frac{1}{x^2}\right)^2 - 1} \qquad \text{Let } x + \frac{1}{x} = t$$

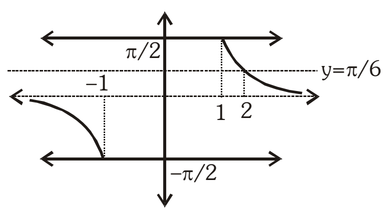
$$= \int \frac{dt}{(t^2-1)} = \frac{1}{2}\ell n\left(\frac{t-1}{t+1}\right) = \frac{1}{2}\ell n\left(\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right)$$

$$= \frac{1}{2}\ell n\left(\frac{x^2-x+1}{x^2+x+1}\right) + c$$

150. (A) $(\operatorname{cosec}^{-1} x)^2 - \left(\dfrac{\pi}{6} + 2\right)\operatorname{cosec}^{-1} x + \dfrac{\pi}{6}.2 \ge 0$

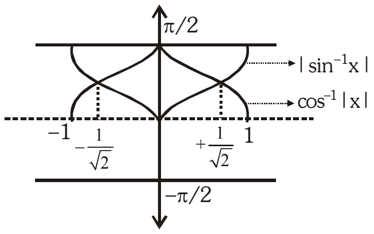
$\Rightarrow (\operatorname{cosec}^{-1} x - 2)(\operatorname{cosec}^{-1} x - \frac{\pi}{6}) \ge 0$

$\Rightarrow \operatorname{cosec}^{-1} x \le \dfrac{\pi}{6}$



$\Rightarrow (-\infty, -1] \cup [2, \infty)$.

(B)



(C) $f(x) = \dfrac{1}{\sqrt{\ell n\left(\dfrac{(\sin^{-1}x).4}{\pi}\right)}}$

$= \ell n\left(\dfrac{(\sin^{-1}x)4}{\pi}\right) > 0 \Rightarrow \sin^{-1}x > \dfrac{\pi}{4}$

$\therefore \quad x \in \left(\dfrac{1}{\sqrt{2}},1\right) \quad x > \dfrac{1}{\sqrt{2}}$

(D) $f(x) = \sqrt{\sin^{-1}(\sin(\sin^{-1}(\sin(\sin^{-1}\sin x))))}$

Domain is subset of [–1, 1]

and $\{\because \sin\sin^{-1}x = x \ \forall -1 \le x \le 1\}$

$\dfrac{\pi}{2} \ge \sin^{-1}(\sin(\sin(\sin^{-1}(\sin x)))) \ge 0$

$1 \ge \sin(\sin^{-1}(\sin(\sin^{-1}(\sin x)))) \ge 0$

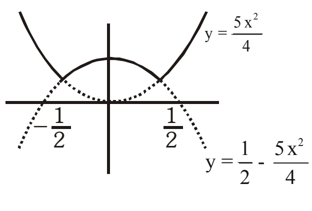
and so on $\Rightarrow 1 \ge \sin x \ge 0$

$\Rightarrow \quad x \in [0, \pi/2]$ also $x \in [-1, 1]$

so $\quad x \in [0, 1]$.

151. (A) $\dfrac{1}{2} - \dfrac{3x^2}{4} = \dfrac{5x^2}{4}$

$\Rightarrow \dfrac{1}{2} = 2x^2 \Rightarrow x = \pm\dfrac{1}{2}$



$f_{min}(x) = f\left(\dfrac{1}{2}\right) = \dfrac{5}{16}$

(B) $f(x + y) = f(x) . f(y) \qquad ...(1)$

put x = y = 0

$\Rightarrow f(0) = f^2(0) \Rightarrow f(0) = 1, 0$

put y = 0 in (1)

$f(x) = f(x) . f(0) \Rightarrow f(x) = 0$ not possible

so $f(0) = 1$

put y = – x in (1)

$\Rightarrow f(0) = f(x) f(-x)$

$f(x) f(-x) = 1$.

(C) $2 . \sin^{-1}(x + 2) = \cos^{-1}(x + 3)$

$\cos(2\sin^{-1}(x + 2)) = x + 3$

$1 - 2(\sin(\sin^{-1}(x + 2)))^2 = x + 3$

$1 - 2(x + 2)^2 = x + 3$

$1 - 2(x^2 + 4x + 4) = x + 3$

$2x^2 + 9x + 10 = 0$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \dfrac{81}{4} - \dfrac{20}{2} = \dfrac{41}{4}$

(D) As P(x) and Q Q(P(Q(x))) have some roots

$\therefore$ Degree of P(x) & Q(P(Q(x))) must be same.

Let the degree of Q(x) is n then degree of P(Q(x)) = 8n

and Q (P(Q(x))) have $8n^2$ so $8n^2 = 8$

$\Rightarrow$ n = 1.

$\Rightarrow$ degree of Q(x) is 1.

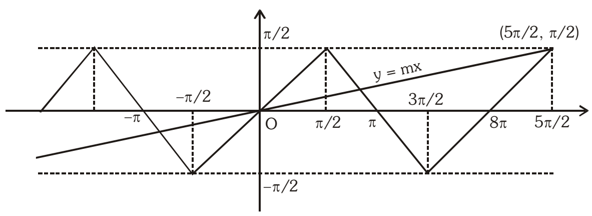
153. (B) $f(x) = \left(\sec^2\dfrac{\pi x}{10} - \tan^2\dfrac{\pi x}{10}\right)^{\cos^4 4\pi x + 100x - \{100x\}}$

Function	Period
$\sec^2\left(\dfrac{\pi x}{10}\right)$	10
$\tan^2\left(\dfrac{\pi x}{10}\right)$	10

$f(x) = 1 \qquad \{\because \ \sec^2\theta - \tan^2\theta = 1\}$

Period = 10

(C) $m = \dfrac{\pi/2 - 0}{5\pi/2 - 0} = \dfrac{1}{5}$



$70m = 70 \times \dfrac{1}{5} = 14$

(D) $f(x) = \sum\limits_{r=1}^{n}\left[r^2 + e^{-x} + r - 1\right]$

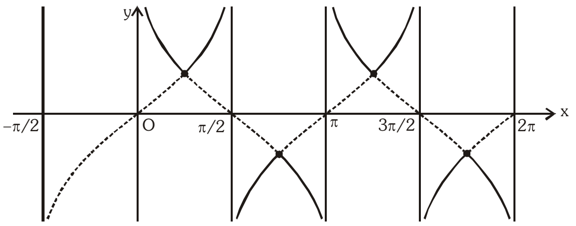
$= \sum\limits_{r=1}^{n}\left(r^2 + r + 1\right) + \left[e^{-x}\right]$

$0 < e^{-x} < 1 \quad \Rightarrow \quad [e^{-x}] = 0$

$\therefore \quad f(x) = \text{constant}$
fundamental period not defined.

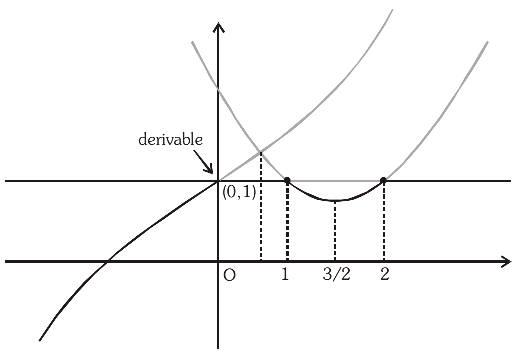
154. (A) $f(x) = \dfrac{\tan x + \cot x}{2} - \left|\dfrac{\tan x - \cot x}{2}\right|$

$f(x) = \begin{cases} \cot x\ , & \tan x \geq \cot x \\ \tan x\ , & \tan x < \cot x \end{cases}$



There are 4 points where the function is continuous but not differentiable in $(0, 2\pi)$

(B)



155. (A) Domain of $f(x)$ is $[-2, 2]$

$f(x)|_{min} = 0 + 4 - 8 + 1$

$= -3 \quad$ at $x = 2$

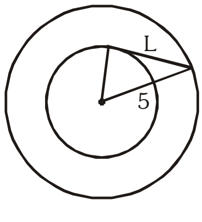
$f(x)|_{max} = \pi + 4 + 8 + 1$

$= \pi + 13 \quad$ at $x = -2$

$\therefore$ Range is $[-3, \pi + 13]$

$a + b = -3 + 13 = 10$

(B)



$L = \sqrt{25 - 21} = 2$

(C) $\displaystyle\int_0^6 e^{\{3x\}}.dx = \int_0^{6.3.\frac{1}{3}} e^{\{3x\}}dx$

$\displaystyle = 18\int_0^{1/3} e^{3x}dx = 18\left(\frac{e^{3x}}{3}\right)_0^{1/3} = 6(e - 1)$

$\therefore \quad p + q = 6 + 1 = 7$

(D) $x^2 - (1 + b)x + b - 2 = 0$

$f(1) = 1 - 1 - b + b - 2 = -2$

$\therefore \quad f(x) = 0$ has atleast one positive root.

157. (A) $\displaystyle I = \int_{-1}^{1} \frac{dx}{(1+x^2)(3^x+1)}$

$\displaystyle I = \int_{-1}^{1} \frac{dx}{(1+x^2)\left(\frac{1+3^x}{3^x}\right)}$

$\displaystyle 2I = \int_{-1}^{1} \frac{3^x + 1}{(1+x^2)(3^x+1)} = 2\int_0^1 \frac{dx}{1+x^2}$

$2I = 2\left(\tan^{-1}x\right)_0^1$

$I = \dfrac{\pi}{4}$

(B) $\displaystyle \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{2.r + n}{r^2 + nr + n^2}\right)$

$\displaystyle \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n^2} n \frac{\left(2.\frac{r}{n} + 1\right)}{\left(\frac{r^2}{n^2} + \frac{r}{n} + 1\right)}$

$\displaystyle \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{2\frac{r}{n} + 1}{\frac{r^2}{n^2} + \frac{r}{n} + 1}\right) \frac{1}{n}$

$\displaystyle \int_0^1 \frac{2x+1}{x^2+x+1}dx = \left(\ell n(x^2 + x + 1)\right)_0^1 = \ell n3$

(C) $\displaystyle \lim_{x\to 0} \frac{\pi \sin^{-1}|x|.1}{4x}$

$\displaystyle \lim_{x\to 0^-} -\frac{\pi \sin^{-1}x}{4x} = -\frac{\pi}{4}$

$\displaystyle \lim_{x\to 0^+} \frac{\pi \sin^{-1}x}{4x} = \frac{\pi}{4}$

$\therefore$ limit does not exist

(D) $\underbrace{\sqrt{\sec^{-1}\left(x + \frac{1}{x}\right)}}_{\text{always positive}}(8x^2 + 2\pi x - \pi^2) \leq 0$

$\therefore \quad (4x - \pi)(2x + \pi) \leq 0$

$x \in \left[-\frac{\pi}{2}, \frac{\pi}{4}\right] - \{0\}$

$\therefore \quad a + b + c = -\frac{\pi}{4}$

159. (A) $I = \int \frac{2(x-1)^2 + x + 1}{x(x-1)^2} dx$

$= 2\int \frac{dx}{x} + \int \frac{dx}{(x-1)^2} + \int \frac{1}{(x-1)} \left(\frac{1}{x-1} - \frac{1}{x}\right) dx$

$= 2\int \frac{dx}{x} + 2\int \frac{dx}{(x-1)^2} - \int \left(\frac{1}{x-1} - \frac{1}{x}\right) dx$

$= 3\int \frac{dx}{x} + 2\int \frac{dx}{(x-1)^2} - \int \frac{dx}{(x-1)}$

$= 3\ell n|x| - \frac{2}{(x-1)} - \ell n|x-1| + C$

$a = 3, b = 2 \Rightarrow a + b = 5$

(B) $\tan\left\{\sin^{-1}\sqrt{\frac{1-x}{2}} + \cos^{-1}\sqrt{\frac{1+x}{2}}\right\}$

$= \sin \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1} x \Rightarrow 0 \le \theta \le \pi$

$\Rightarrow \tan\left\{\sin^{-1}\sqrt{\frac{1-\cos\theta}{2}} + \cos^{-1}\sqrt{\frac{1+\cos\theta}{2}}\right\}$

$= \frac{2}{\sqrt{5}} \Rightarrow \tan\left\{\frac{\theta}{2} + \frac{\theta}{2}\right\} = \frac{2}{\sqrt{5}}$

$\Rightarrow \tan(\cos^{-1}x) = \frac{2}{5}$

$\Rightarrow \tan\left(\tan^{-1}\frac{\sqrt{1-x^2}}{x}\right) = \frac{2}{\sqrt{5}}$

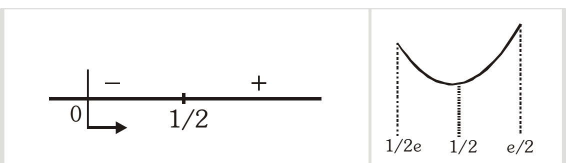
$\Rightarrow \frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}} \Rightarrow 5 - 5x^2 = 4x^2$

$\Rightarrow 9x^2 = 5 \quad \Rightarrow \quad \frac{27x^2}{5} = 3$

(C) $f(x) = (x\ell n2x - x)$

$f'(x) = \ell n2x + 1 - 1$

$f'(x) = \ell n2x$



$f\left(\frac{1}{2}\right) = -\frac{1}{2}, f\left(\frac{e}{2}\right) = \left(\frac{e}{2}\right) \ell ne - \frac{e}{2} = 0$

Range $\left[-\frac{1}{2}, 0\right] \Rightarrow a + b = 2$

(D) $I = \int \tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right) dx$

$x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$I = \int \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) \sec^2\theta d\theta$

$= \int \tan^{-1}\left(\tan\frac{\theta}{2}\right) \sec^2\theta d\theta$

$\Rightarrow \quad 2I = \int \theta \sec^2\theta d\theta$

$= \theta\tan\theta - \int \tan\theta d\theta = \theta\tan\theta - \ell n|\sec\theta| + C$

$\therefore \quad I = \frac{1}{2} x.\tan^{-1}x - \frac{1}{2} \ell n\left|\sqrt{x^2+1}\right| + C$

$= \frac{1}{2} x \tan^{-1} x - \frac{1}{4} \ell n(x^2+1) + c$

$a + b = 6$

161. (A) $y = \lim_{x\to 0}\left(\int_0^x \sin t^2 dt\right)^{\tan x} = (0)^0$ from

$\ell ny = \lim_{x\to 0} \tan x\left(\ell n\left(\int_0^x \sin t^2 dt\right)\right)$

$= \lim_{x\to 0} \frac{\ell n\left(\int_0^x \sin t^2 dt\right)}{\cot x} = \lim_{x\to 0} \frac{\sin x^2}{\left(\int_0^x \sin t^2 dt\right)(-\text{cosec}^2 x)}$

$= -\lim_{x\to 0} \frac{\sin x^2 \sin^2 x}{\int_0^x \sin t^2 dt} = \lim_{x\to 0} \frac{x^4}{\int_0^x \sin t^2 dt}$

$= \lim_{x\to 0} \frac{4x^3}{\sin x^2} = 0 \Rightarrow y = 1$

$\pi\ell = \pi$

(B) $0 < x < \frac{\pi}{2} \Rightarrow f(x) = \lim_{n\to\infty} \frac{\left(\frac{x}{\pi/2}\right)^n + 1}{\left(\frac{x}{\pi/2}\right)^{n-1} + 1}\frac{\pi}{2} = \frac{\pi}{2}$

$\frac{\pi}{2} < x < \pi$

$\Rightarrow f(x) = \lim_{n\to\infty}\left(\frac{1+\left(\frac{\pi/2}{x}\right)^n}{1+\left(\frac{\pi/2}{x}\right)^{n-1}}\right)x = x$

Now $\int_0^{\pi/2}\frac{\pi}{2}\cos dx + \int_{\pi/2}^{\pi} x\cos x dx$

(C) $\int_0^{\pi/2}\frac{\sin\theta\cos\theta d\theta}{(\sin\theta+\cos\theta+1)} = -\frac{1}{2}\int_0^{\pi/2}\frac{\sin 2\theta d\theta}{(\sin\theta+\cos\theta)+1} - 1$

$= \frac{1}{2}\int_0^{\pi/2}\frac{(\sin\theta+\cos\theta)^2-1}{(\sin\theta+\cos\theta+1)}d\theta - 1$

$= \frac{1}{2}\int_0^{\pi/2}(\sin\theta+\cos\theta-1)d\theta - 1$

(D) $\frac{32.4}{5}\int_0^{\pi}\cos^6 2x\sin^2 2x dx$

let $2x = t \qquad dx = \frac{dt}{2}$

$= \frac{64}{5}\int_0^{2\pi}\sin^2 t\cos^6 t dt = 64.4\int_0^{\pi/2}\sin^2 t\cos^6 t dt$

Now apply wallis theorem

165. $\left.\begin{matrix}6x+2x^2 = 2ax+bx^2\\ 5x+4x^2+3 = 5x+cx^2+3\end{matrix}\right\} \Rightarrow a = 3, b = 2 \ \&\ c = 4$

Given functional relation is

$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$

$x = y = 0,\ f(0) = 0$

$y = -x$

$f(x) + f(-x) = f(0)$

$f(x) = -f(-x)$

Now, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{f(x+h)+f(-x)}{h}$

$= \lim_{h\to 0}\frac{f\left(\frac{x+h-x}{1-(x+h)(-x)}\right)}{h} = \lim_{h\to 0}\frac{f\left(\frac{h}{1+x(x+h)}\right)}{h}$

$= \lim_{h\to 0}\frac{f\left(\frac{h}{1+x(x+h)}\right)}{\left(\frac{h}{1+x(x+h)}\right)}\times\left(\frac{1}{1+x(x+h)}\right)$

$f'(x) = \frac{2}{1+x^2} \qquad \left\{\because \lim_{x\to 0}\frac{f(x)}{x} = 2\right\}$

$f(x) = 2\tan^{-1}x + c$

$f(0) = 0 + c \Rightarrow c = 0 \qquad \{\because f(0) = 0\}$

$\therefore\ f(x) = 2\tan^{-1}x$

$\int_0^1\frac{ax^2+bx+c}{f(1)}dx = \int_0^1\frac{3x^2+2x+4}{2.\pi/4}dx$

$= \frac{2}{\pi}(1+1+4) = \frac{12}{\pi}$

$\therefore \left[\frac{12}{\pi}\right] = 3$

166. $\int_0^a f(g(x))dx = 1 - \frac{e^{-2a}}{2} \Rightarrow f(g(a)) = e^{-2a}$

Given $\frac{xd\{f(g(x))\}}{f(g(x))} = \frac{d\{g(f(x))\}}{g(f(x))}$

$\Rightarrow \frac{x(-2e^{-2x})dx}{e^{-2x}} = \frac{d\{g(f(x))\}}{g(f(x))}$

or $-2xdx = \frac{d\{g(f(x))\}}{g(f(x))}$

$\Rightarrow -x^2 = \ln\{g(f(x))\} + c$

$\Rightarrow g(f(x)) = e^{-x^2}$ as $g(f(0)) = 1$

$\Rightarrow |\ln\{g(f(4))\}| = 16$