JMO - 2019

Duration: 3 Hrs. **Maximum Marks : 100**

1. If the counting numbers x, y, z represent three sides of any triangle and y < z, then find the maximum number of triangles possible.

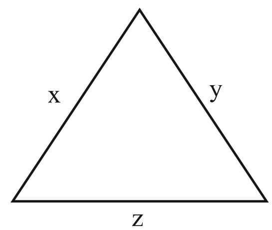
Sol. Given: y < z ...(1)

Using inequality in triangle

$y+z>x$ ...(2)

$x + y > z$

$x > z - y$ ...(3)



So, since x, y, z are natural numbers, $x \geq 1$

So, x can have values

$x-y+1,\ z-y+2...,\ y+z-1$

This is an AP with common difference 1 and first terms being z – y + 1. So, sum of all values should be maximum number of possible triangles.

$N=(y+z-1)-(z-y+1)+1$

$= 2y - 1$

2. Find the sum of different 10th place digits of all perfect square numbers whose unit's place digits are perfect square.

Sol. Since numbers are perfect squars, they wil end with 0, 1, 4, 5, 6, 7.

Now, since unit digit is also a perfect square. So numbers should be ending with 0, 1, 4, 9.

Possible values are :

$7^2 = 49$

$8^2 = 64$

$9^2 = 81$

$10^2 = 100$

$11^2 = 121$

$12^2 = 144$

$13^2 = 169$

$17^2 = 289$

$18^2 = 324$

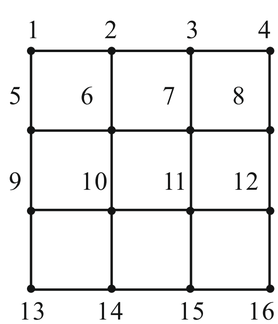
$19^2 = 361$

$20^2 = 400$

Now, we observe that tens digit is an even number. So, sum of all even numbers is 0+2+4+6+8=20

3. Arrange 16 plants in 8 rows with 4 plants in each row.

Sol.



4. Find the remainder when 2^{2019} is divided by 2019.

Sol. $\frac{2^{2019}}{2019}$

$=\frac{2^{2019}}{3\times 673}$

If we divide 2^{2019} by 3, remainder will be 2.

If we divide 2^{2019} by 673, remainder will be 8

Now, 2^{2019} = 673 m + 8 & 2^{2019} = 3n + 2

$\therefore \quad 3n+2=673m+8$

$\Rightarrow n=\frac{673m+6}{3}=\frac{673}{3}m+2$

$\therefore \quad m=3,6,9,.....$

$\therefore \quad m=3k$

$2^{2019}=673\ (3k)+8$

$=2019k+8$

$\therefore$ Remainder in division = 8

5. If x + y + z = 6,

$x^2 + y^2 + z^2 = 26$,

$x^3 + y^3 + z^3 = 90$,

then find $x^4+y^4+z^4$

Sol. $x+y+z=6$

$(x+y+z)^2=36$

$x^2+y^2+z^2+2\ (xy+yz+zx)=36$

$(xy+yz+zx)=\frac{36-26}{2}=5$

$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$90-3xyz=6(26-5)$

$3xyz=90-126$

$xyz=-12$

$(x^2+y^2+z^2)^2=x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2$

$(261^2-2\left[(xy+yz+zx)^2-2xyz(x+y+z)\right]=x^4+y^4+z^4$

$676-2[25+24.6]=x^4+y^4+z^4$

$676-338=x^4+y^4+z^4$

$x^4+y^4+z^4=338$

6. Find the sum $\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+....+\frac{50}{1+50^2+50^4}$

Sol. $\displaystyle\sum_{n=1}^{50}\frac{n}{1+n^2+n^4}=\sum_{n=1}^{50}\frac{n}{(1+n+n^2)\,(n^2-n+1)}$

$$=\sum_{n=1}^{50}\frac{1}{2}\left[\frac{1}{(n^2-n+1)}-\frac{1}{n^2+n+1}\right]$$

$$=\frac{1}{2}\sum_{n=1}^{50}\frac{1}{n^2+n+1}-\frac{1}{n^2+n+1}$$

$$=\frac{1}{2}\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{3}-\frac{1}{7}+\frac{1}{7}-\frac{1}{13}....\frac{1}{2551}\right]$$

$$=\frac{1}{2}\left(1-\frac{1}{2551}\right)$$

$$=\frac{1}{2}.\frac{2550}{2551}=\frac{1275}{2551}$$

7. If $E=\left(\frac{1}{81}\right)^{-16^{-4^{-2^{-1}}}}+(25)^{-8^{-3^{-1}}}$, then determine E.

Sol. $E=\left(\frac{1}{81}\right)^{-16^{-4^{-2^{-1}}}}+(25)^{-8^{-3^{-1}}}$

$$=(81)^{16^{-4^{-2^{-1}}}}+\left(\frac{1}{25}\right)^{8^{-3^{-1}}}$$

$$=(81)^{\frac{1}{16}^{\frac{1}{4}^{\frac{1}{2}}}}+\left(\frac{1}{25}\right)^{\frac{1}{8}^{\frac{1}{3}}}$$

$$=(81)^{\frac{1}{4}}+\left(\frac{1}{25}\right)^{\frac{1}{2}}$$

$$=3+\frac{1}{5}=\frac{16}{5}$$

8. Find the positive 4 digit number whose value is increased by exactly 60% when the 1st digit from the left is shifted to the end of the number.

Sol. Let number be abcd.
So, $10^3a + 10^2b + 10c + d$ is expanded form of number.
When number at left end is shifted to right end then number will be
bcda = $10^3b + 10^2c + 10d + a$
Now, bcda = 160% of abcd

$$\left(10^3b+10^2c+10d+a\right)=\frac{8}{5}\left(10^3a+10^2b+10c+d\right)$$

$$5000b+500c+50d+5a=8000a+800b+80c+8d$$

$$7995a=42(100b+10c+d)$$

$7995a=42(bcd)$ 3-digit number

2665 a = 14 (bcd)

Since 14 & 2665 are co-prime a has to be 14. Which is not possible as a is a single-digit number. So, there is no such four digit number.

9. Between 5'O clock and 6'O clock a man looked at his watch and by mistake he read the hour hand as minute hand and vice virsa. As a result he observed that the time was 57 minutes earlier than the correct time. Determine the correct time.

Sol. It is not possible as in such cases the difference between consecutive hours is always $55\frac{5}{13}$ minutes.

10. In a 100 m race, Ramesh defeated Shyam by 15 m. In another competition of a similar race Harish defeated Shyam by 20 m. If Ramesh and Harish take part in 150 m race, then find who will win the race and by what distance.

Sol. R S
100 85
Ratio of speed of R : S = 20 : 17
S H
80 100
Ratio of speed of S : H = 4 : 5

$\frac{R}{S}:\frac{H}{S}=\frac{R}{H}$

$\frac{20}{17}\times\frac{4}{5}=\frac{R}{H}$

$\frac{R}{H}=\frac{20}{85}\Rightarrow\frac{R}{H}=\frac{16}{17}$

So, if Harish travels 150 m. Ramesh will travel $\frac{16}{17}150=141\frac{3}{17}$m

So, Harish will defeat Ramesh by $8\frac{14}{17}$m

11. A and B have some marbles with them. When A gives 100 marbles to B than the number of marbles with B becomes double that of A. However when B gives some marbles to A, then A will have 5 times more marbles than B. What was the minimum possible number of marbles with A originally and how many marbles were there with B is such case?

Sol. $2(A-100)=B+100$

$2A-B=300$ ...(1)

Let B gives m marbles to A.

$5(B-m)=(A+m)$

$5B-A=6m$

$12\text{ m}=10B-2A$

$12\text{ m}=9B-(2A-B)$

$12\text{ m}=9B-300$

$4\text{ m}=3B-100$

Now, 3B > 100 since m is +ve number and 3B – 100 should be a multiple of 4 as m is integral.
Hence, 3B = 108 (least possible value)
B = 36

$A\Rightarrow 168$ and $m=2$

12. Five men A, B, C, D and E are wearing caps of black or white colour without knowing the colour of their own caps. It is known that a man wearing a black cap always speaks the truth while the one wearing white always tells lies. If they make the following statements, find the colour worn by each of them.

A : I see three black caps and one white cap

B : I see four white caps

C : I see one black cap and three white caps

D : I see four black caps

E : No statements

Sol. Case (1) A is telling the truth

A $\rightarrow$ Black
B $\rightarrow$ White (lying)
C $\rightarrow$ Can't be determined
D $\rightarrow$ White (lying)
E $\rightarrow$ Can't be determined

Case (2) C is telling the truth

A $\rightarrow$ White (lying)
B $\rightarrow$ White (lying)
C $\rightarrow$ Black
D $\rightarrow$ White (lying)
E $\rightarrow$ Black (By statement of C)

So, A $\rightarrow$ White , B $\rightarrow$ White, C $\rightarrow$ Black, D $\rightarrow$ White, E $\rightarrow$ Black.

13. Two women went to a market to sell 100 eggs. Out of 100 eggs one of them took some eggs and went to a different place to sell. After selling all the eggs each received the same amount. Then the first woman told; had I sold your eggs I would have got 150 rupees. The second woman told; had I sold your eggs I would have got 66 2/3 rupees. Determine how many eggs each woman sold.

Sol. Let R_1 is rate at which first woman sold and R_2 is rate of which second woman sold.
Let first woman sold x eggs.

So, $R_1x = R_2(100-x)$

$$\frac{R_1}{R_2} = \frac{100-x}{x} \quad ...(1)$$

Now, $R_1(100-x) = 150$ ...(2)

$$R_2.x = \frac{200}{3} \quad ...(3)$$

(2) : (3)

$$\frac{R_1}{R_2}\frac{(100-x)}{x} = \frac{150}{200}\times 3$$

$$\frac{(100-x)}{x}\frac{(100-x)}{x} = \frac{3}{4}\times 3$$

$$\frac{x^2}{(100-x)^2} = \frac{4}{9}$$

$$\frac{x}{100-x} = \frac{2}{3}$$

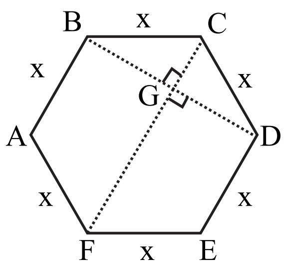
So, x = 40
100 – x = 60

14. ABCD is a quadrilateral in which $\angle DAB + \angle ABC = 90°$. If $BC = 9$ cm, $BD = 6$ cm and $AC = 7$ cm, then find the area of the square CDEF.

Sol. Data inadequate

15. ABCDEF is a regular hexagon. Diagonals FC and BD intersect each other at G. Determine the ratio of the area of the triangle BCG and the area of the quadrilateral DEFG.

Sol. Let side of Hexagon is x.



Area of FEDC $= \frac{1}{2}$ Area of hexagon

$$= \frac{1}{2}\left[\frac{3\sqrt{3}}{2}x^2\right]$$

$$= \frac{3\sqrt{3}}{4}x^2$$

Area of $\Delta GBC = \frac{1}{2}.OB.OC$

$$= \frac{1}{2}\left[\frac{\sqrt{3}x^2}{4}\right] = \frac{\sqrt{3}x^2}{8}$$

$$\frac{\text{Area of GBC}}{\text{Area of DEFG}} = \frac{\frac{\sqrt{3}x^2}{8}}{\frac{3\sqrt{3}}{4}x^2 - \frac{\sqrt{3}}{8}x^2} = \frac{\frac{\sqrt{3}}{8}}{\frac{5\sqrt{3}}{8}}$$

$$\frac{\text{Area of GBC}}{\text{Area of DEPG}} = \frac{1}{5}$$

16. In the square ABCD, 'P' is one point inside the square such that PA = 15 cm, PC = 20 cm, PD = 7 cm. Determine PB.

Sol. $PA^2 + PC^2 = PB^2 + PD^2$

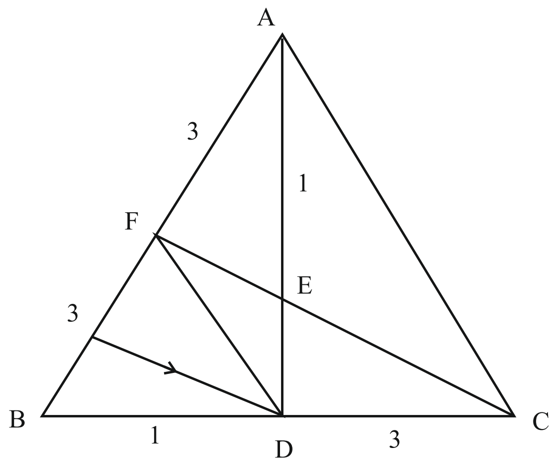
$15^2 + 20^2 = PB^2 + 7^2$

$625 - 49 = PB^2$

PB = 24 cm

17. Area of triangle ABC is 14 cm^2 D is a point on BC such that CD = 3 BD. E is the mid point of AD. CE when produced meets AB at F. Find the sum of the area of the triangles CDE and AEF.

Sol.



Area of triangle = 14 cm^2

Let ara of ΔABD be x cm^2

$\therefore$ Area of Δ ADC $= 3x$ cm^2

Area of Δ ABD + Area of ΔADC $= 14$

$4x = 14$

$x = \frac{14}{4} = \frac{7}{2}$

$\therefore$ Area of $\Delta ADC = 3x = \frac{21}{2}$

$\therefore$ Area of $\Delta CDE = \frac{21}{2} \times \frac{1}{2} = \frac{21}{4} cm^2$

Constant DG parallel to CF

In ΔFGB

$\frac{FG}{GB} = \frac{CD}{BD}$ [by BPT]

$\frac{FG}{GB} = \frac{3}{1}$

In ΔAGD

$\frac{AF}{FG} = \frac{AE}{ED}$

$\frac{AF}{FG} = \frac{1}{1}$

$\frac{AF}{3} = 1$

AF = 3

Now, jon DF,

$\therefore$ Area of $\Delta DBG : \Delta DFG : \Delta DFA$

1 : 3 : 3

y : 3y : 3y

$y+3y+3y=\frac{7}{2}$

$7y=\frac{x}{2}$

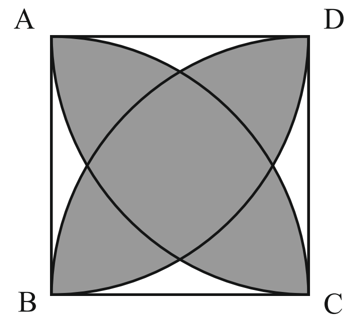
$y=\frac{1}{2}$

$\therefore$ Area of $\Delta AFE+$ Area of ΔCED

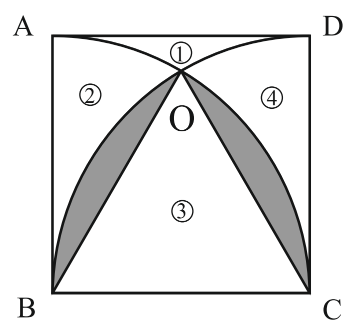
$\frac{3}{4}+\frac{21}{4}$ = 6 cm^2

18. In the adjoining figure ABCD is a square of side 10 cm each. Quadrants are drawn with each vertex of the square as centere. Find the area of the shaded portion

(Take $\sqrt{3}$ = 1.1732 and π = 3.14)



Sol. Now, join BO and CO, so OBC become equilateral triangle.



$\therefore$ Area of sector BOCB $=\frac{60}{360}\times\pi r^2=\frac{\pi r^2}{6}$

$\therefore$ Area of shaded region $=\frac{\pi r^2}{6}-\frac{\sqrt{3}}{4}r^2$

$=\frac{\pi.100}{6}-\frac{\sqrt{3}}{4}.100$

$=100\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)$

$\therefore$ Area of region (2) $= \frac{\pi r^2}{4} - \frac{\sqrt{3}}{4} r^2 - 100\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) \times 2$

$$= \frac{\pi}{4} \times 100 - \frac{\sqrt{3}}{4} \times 100 - 200\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

$$= 100\left[\frac{\pi}{4} - \frac{\sqrt{3}}{4} - \frac{2\pi}{6} + \frac{2\sqrt{3}}{4}\right]$$

$$= 100\left[\frac{3\pi - 3\sqrt{3} - 4\pi + 6\sqrt{3}}{12}\right]$$

$$= 100\left[\frac{3\sqrt{3} - \pi}{12}\right]$$

$\therefore$ Area of region (1) = Area of square – region (2+3+4)

$$= 100 - 100\left[\frac{3\sqrt{3} - \pi}{12}\right] \times 2 - 100\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) \times 2 - \frac{\sqrt{3}}{4} \times 100$$

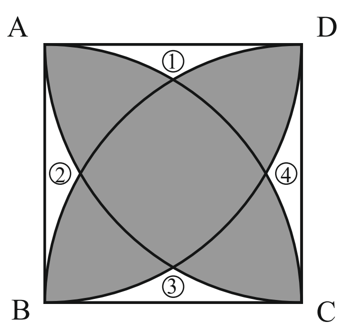
$$= 100 - \frac{100}{6}\left[3\sqrt{3} - \pi\right] - 100\left(\frac{2\pi - 3\sqrt{3}}{12}\right) \times 2 - 25\sqrt{3}$$

$$= \frac{600 - 300\sqrt{3} + 100\pi - 200\alpha + 300\sqrt{3} - 150\sqrt{3}}{6}$$

$$= \frac{600 - 150\sqrt{3} - 100\pi}{6}$$

as the figure is symmetric.,

$\therefore$ Area of region (1 + 2 + 3 + 4)
= 4 × Area of region (1)



$$= 4\left(\frac{600 - 150\sqrt{3} - 100\pi}{6}\right)$$

$$= \frac{2}{3}\left(600 - 150\sqrt{3} - 100\pi\right)$$

$\therefore$ Area of shaded regions = Area of square – Area of region (1 + 2 + 3 + 4)

$$= 100 - \left[\frac{2}{3}(600 - 150\sqrt{3} - 100\pi\right]$$

$$=100-\left[400-100\sqrt{3}-\frac{200\pi}{3}\right]$$

$$=100-400+100\sqrt{3}+\frac{200\pi}{3}$$

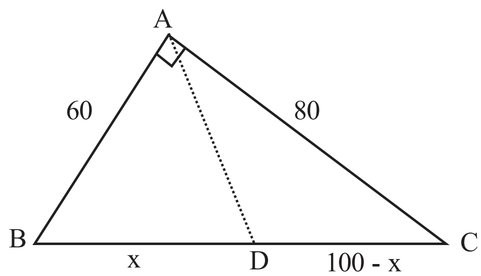
$$=\frac{200\pi}{3}+100\sqrt{3}-300$$

$$=209.33+173.2-300$$

$$=82.53\ \text{cm}^2$$

19. In the ΔABC, $AB=60$ cm, $AC=80$ cm, $BC=100$ cm and D is a point on BC such that the perimeter of the ΔABD and the perimeter of the ΔACD are equal. Find AD.

Sol. AB = 60
AC = 80
BC = 100

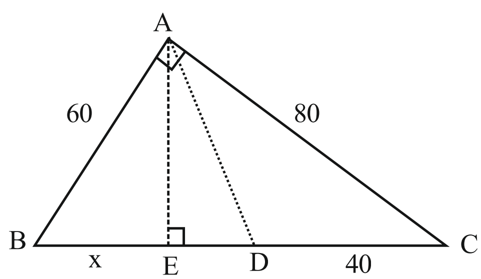


$AD+x+60=80+AD+100-x$

$2x=120$

$x=60$

$BD=60,\ \ CD=40$



$$\frac{1}{2}AE.100=\frac{1}{2}.60.80$$

$AE=48$

$BE=\sqrt{60^2-48^2}$

BE = 36

DE = 60 – 36 = 24

$AD^2 = AE^2 + DE^2$

$AD^2=48^2+24^2$

$AD^2=24^2(2^2+1)$

$AD = 24\sqrt{5}$ cm

20. Solve :

$$\frac{1}{x}+\frac{1}{y+z}=\frac{1}{2}$$

$$\frac{1}{y}+\frac{1}{z+x}=\frac{1}{3}$$

$$\frac{1}{z}+\frac{1}{x+y}=\frac{1}{4}$$

Sol. $\frac{1}{x}+\frac{1}{y+z}=\frac{1}{2}$

$$\frac{1}{y+z}=\frac{1}{x}-\frac{1}{x}$$

$$y+z=\frac{2x}{x-2}$$

$xy + xz = 2(x+y+z)$ ...(1)

Similarly,

$zy + xy = 3(x+y+z)$ ...(2)

$xz + yz = 4(x+y+z)$ ...(3)

(1) & (2)

$$zy + 2(x+y+z) - xz = 3(x+y+z)$$

$$z(y-x) = (x+y+z)$$

from (3), $z(y-x)=\frac{z(x+y)}{4}$

$$4y-4x = x+y$$

$$3y = 5x$$

$$\frac{x}{y}=\frac{3}{5}$$

(3) - (2) $\Rightarrow x(z-y) = x+y+z$

from (1),

$$x(z-y)=\frac{x(y+z)}{2}$$

$$2x-2y = y+z$$

$$z = 3y$$

$$\frac{y}{z}=\frac{1}{3}$$

So, $\frac{x}{2}=\frac{x}{y}.\frac{y}{z}=\frac{3}{5}.\frac{1}{3}=\frac{1}{5}$

x : y : z
3 : 5 :15

So, ratio is z : y : z is 3 : 5 : 15.
Now lets take a as multiplying factor.
So, x = 3a, y = 5a, z = 15a.

(1) $\Rightarrow \frac{1}{3a}+\frac{1}{5a+15a}=\frac{1}{2}$

$$\frac{23a}{60a^2}=\frac{1}{2}$$

$$a=\frac{23}{30}$$

So, $x=\frac{23}{10},\ y=\frac{23}{6},\ z=\frac{23}{2}$