ALLEN CAREER INSTITUTE KOTA (RAJASTHAN) Path to success

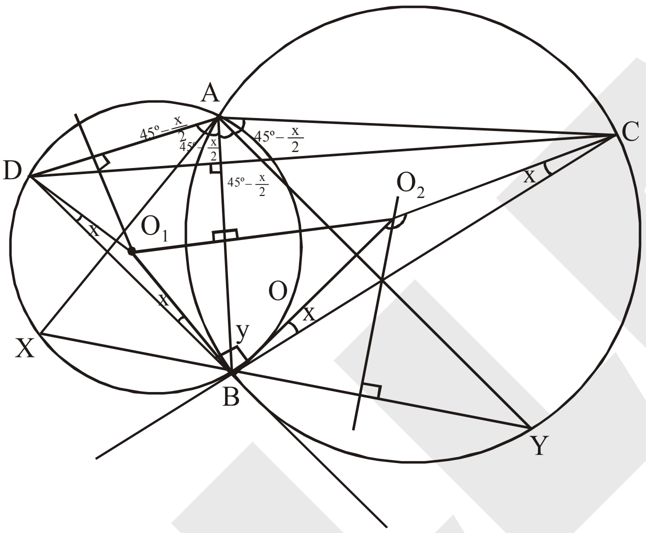
35th Indian National Mathematical Olympiad-2020

Date of Examination : 19th January, 2020

SOLUTIONS

1. Let τ_1 and τ_2 be two circles of unequal radii, with centers O_1 and O_2 respectively, in the plane intersecting in two distinct points A and B. Assume that the centre of each of the circles τ_1 and τ_2 is outside the other. The tangent to τ_1 at B intersects τ_2 again in C, different from B, the tangent to τ_2 at B intersects τ_1 again in D, different from B. The bisectors of $\angle DAB$ and $\angle CAB$ meet τ_1 and τ_2 again in X and Y, respectively, different from A. Let P and Q be the circumcenter of triangles ACD and XAY respectively. Prove that PQ is the perpendicular bisector of the segment O_1 and O_2.

Sol.



$\angle O_1BC = 90 = \angle O_1BO_2 + \angle O_2BC$

$\angle O_2BD = 90 = \angle O_2BO_1 + \angle O_1BD$

$\therefore\ \angle OB_1D = \angle O_2BC = x$

$\therefore\ \angle O_1DB = \angle O_1BD = x = \angle O_2BC = \angle O_2CB$

$\therefore\ \angle DO_1B = 180° - 2x = \angle BO_2C$

$\therefore\ \angle DAB = 90° - x = \angle BAC$

Also AX and AY are the angle bisecturs

$\angle DAX = \angle XAB = \angle BAY = \angle YAC$

and $\angle ABD = \angle ACB$ (alternate segment angle)

$\angle ADB = \angle ABC$ (alternate segment angle)

Now we will show that P, Q lie on the $\perp$ bisector of O_1O_2

Since P lies on $\perp$ bisector of AD & AC os $O_1P \perp AD$

Also $O_1O_2 \perp AB$

$\therefore\ \angle PO_1O_2 = 180 - \angle DAB$

$\angle PO_2O_1 = 180 - \angle BAC$

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hence $\angle PO_1O_2 = \angle PO_2O_1$ $[\because \angle BAC = \angle DAB]$

Thus $PO_1 = PO_2$

Similarly Q lies on $\perp$ bisector of AY and AX

i.e. $QO_2 \perp AX$ and $O_1O_2 \perp AB$

we have

$\angle QO_2O_1 = \angle BAY$ and

$\angle QO_1O_2 = \angle BAX$

Since $\angle BAX = \angle BAY$ as done earlier

$\therefore \angle QO_2O_1 = \angle QO_1O_2$

$\therefore QO_1 = QO_2$

Thus P, Q lies on $\perp$ bisector of O_1O_2

Thus PQ is a $\perp$ bisector of O_1O_2

2. Suppose P(x) is a polynomial with real coefficients satisfying the condition.

$P(\cos\theta + \sin\theta) = P(\cos\theta - \sin\theta)$.

for every real θ prove that P(x) can be expressed in the form

$P(x) = a_0 + a_1(1 - x^2)^2 + a_2(1 - x^2)^4 + \ldots\ldots\ldots a_n(1 - x^2)^{2n}$.

for some real numbers $a_0, a_1, a_2 \ldots\ldots a_n$ and nonnegative integer n.

Sol. $P(\cos\theta + \sin\theta) = P(\text{cso}\,\theta - \sin\theta)$

for $\theta = \frac{\pi}{2} \Rightarrow P(1) = P(-1) = c(\text{let})$

so, $(1 - x^2) \mid P(x) - c$

so, $P(x) = c + Q(x)(1 - x^2)$

Now substituting this in original equation we will get $Q(\cos\theta + \sin\theta) = -Q(\cos\theta - \sin\theta)$

Note :- This will hold for $\sin 2\theta \neq 0$ But Q is a polynomial it will hold $\forall\ \theta$.

so $Q(\cos\theta + \sin\theta) = -Q(\cos\theta - \sin\theta)$

Now by substituting $\theta = 0$ & $\theta = \pi$ will get

$Q(1) = Q(-1) = 0$ so $(1 - x^2) \mid Q(x)$

so $Q(x) = (1 - x^2)\, Q'(x)$

so $P(x) = P(1) + (1 - x^2)^2\, Q(x)$

Now again substituting in main equation we will get $Q'(\cos\theta + \sin\theta) = Q'(\cos\theta - \sin\theta)$.

This completes the proof.

3. Let X = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. Let $S \subseteq X$ be such that any positive integer n can be written as p + q where the non-negative integers p, q, have all their digits in S. Find the smallest possible number of elements is S.

Sol. Claim : $|S| \geq 5$

Proof : If possible let $|S| = 4$ and $S = \{\alpha, \beta, \gamma, \delta\} \subset \{0, 1, 2, \ldots\ldots\ldots 9\}$

Using any two elements of S with repetition.

Allowed we must able to set a solution for every $k \in \{0, 1, 2, \ldots\ldots, 9\}$ of the equation

$x + y \equiv k \bmod (10)$ where $x, y \in S$

$\Rightarrow$ x + y should take five times even value and five times odd value.

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Now number of ways to select two elements with repetition allowed from S is $\binom{4}{2}+\binom{4}{1} = 10$

$\Rightarrow$ We can define 10 sums out of which 4 namely $\alpha + \alpha$, $\beta + \beta$, $\gamma + \gamma$, $\delta + \delta$ are definitly even.

As we need five even numbers, we need one more even sum say $\alpha + \beta$ even

$\Rightarrow$ $\alpha \equiv \beta \pmod 2$

If y or $\delta \equiv \alpha \pmod 2$ we will not set five odd sums

If y or $\delta \not\equiv \alpha \pmod 2$ then

$y \equiv \delta \pmod 2 \quad \Rightarrow \quad y + \delta = \text{even}$

$\Rightarrow$ we will not get five odd sums

Hence $|S| \geq 5$

Now one such S can be {0, 1, 2, 5, 8}

Observe :
$0 = 0 + 0$
$1 = 0 + 1$
$2 = 2 + 0$
$3 = 1 + 2$
$4 = 2 + 2$
$5 = 5 + 0$
$6 = 5 + 1$
$7 = 5 + 2$
$8 = 8 + 0$
$9 = 1 + 8$

$\Rightarrow$ Every digit of $x \in \{0, 1, 2, 3 .... 9\}$ can be split as above to set a and b we will remove will leading zeroes in a or b.

4. Let $n \geq 3$ be an integer and let $1 < a_1 \leq a_2 \leq a_3 \leq .... \leq a_n$ be n real numbers such that $a_1 + a_2 + a_3 + ... + a_n = 2n$. Prove that

$$a_1a_2.........a_{n-1} + a_1a_2........a_{n-2} + ... + a_1a_2 + a_1 + 2 \leq a_1a_1........a_n$$

Sol. Let us illustrate it for 4 variables.

We have $a_3a_2a_1 \geq a_2a_1 \geq a_1 > 1$

and $a_4 - 1 \geq a_3 - 1 \geq a_2 - 1 \geq a_1 - 1$

Now, we know that if

$x_i\ y_i \in R\ \forall\ i = i, 2, 3\ .....\ n$ such that

$x_1 \leq x_2 \leq x_3\ .... \leq x_n$ and $y_1 \leq y_2\ .... \leq y_n$, then

By using Chebyshev's inequality, we can write

$$\left(\frac{x_1 + x_2......x_n}{n}\right)\left(\frac{y_1 + y_2......y_n}{n}\right) \leq \left(\frac{x_1y_1 + x_2y_2......x_ny_n}{n}\right)$$

Using Chebyshev's inequality in our problem,

$$\left(\frac{a_3a_2a_1 + a_2a_1 + a_1 + 1}{4}\right)\left(\frac{a_4 - 1 + a_3 - 1 + a_2 - 1 + a_1 - 1}{4}\right) \leq \frac{a_3a_2a_1(a_4 - 1) + a_2a_1(a_3 - 1) + a_1(a_2 - 1) + 1(a_1 - 1)}{4}$$

$\Rightarrow$ $(a_3a_2a_1 + a_2a_1 + a_1 + 1)(a_4 + a_3 + a_2 + a_1 - 4) \le 4(a_4a_3a_2a_1 - 1)$

Now since $a_1 + a_2 + a_3 + a_4 = 2(4) = 8$

So $(a_3a_2a_1 + a_2a_1 + a_1 + 1)\ \cancel{(4)} \le \cancel{(4)}\ (a_4a_3a_2a_1 - 1)$

$\Rightarrow$ $a_3a_2a_1 + a_2a_1 + a_1 + 2 \le a_4a_3a_2a_1$.

Which is the required result. Same way we can show this for n variables using some procedure.

5. Infinitely many equidistant parallel lines are draw in the plane. A positive integer $n \ge 3$ is called frameable if it is possible to draw a regular polygon with n sides all whose vertices lie on these lines and no line contains more than one vertex of the polygon.

(a) Show that 3, 4, 6 are frameable

(b) Show that any integer $n \ge 7$ is not frameable

(c) Determine whether 5 is frameable

Sol. (a) For n = 3, 4, 6 it is possible to draw regular polygons with vertices on the parallel lines.

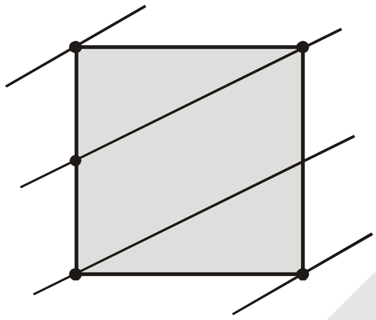

Figure 1

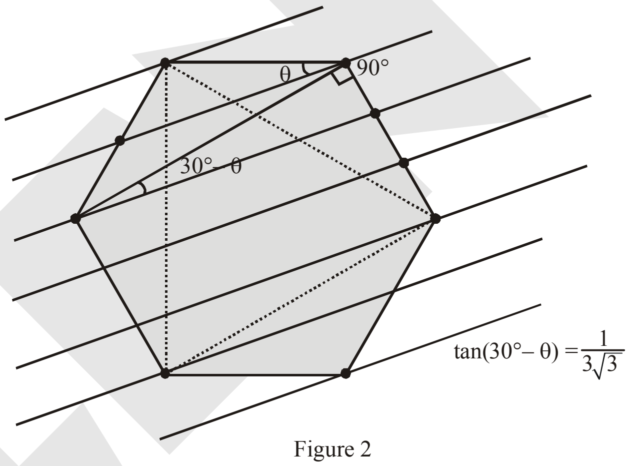

Figure 2

(b) We will prove that it is not possible for $n \ge 7$. In fact, we prove a stronger statement that we can not draw other polygons with vertices on the lines (even if we allow more than one vertex to lie on the same line).

First observe that if A, B are points on the line and C is another point on a line, if we

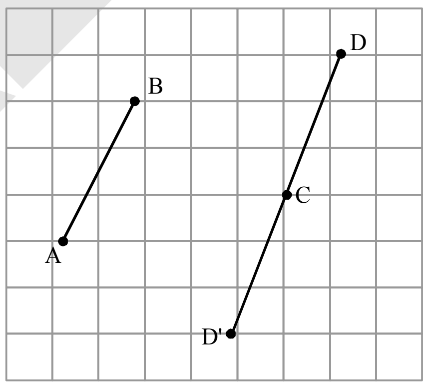

Figure 3

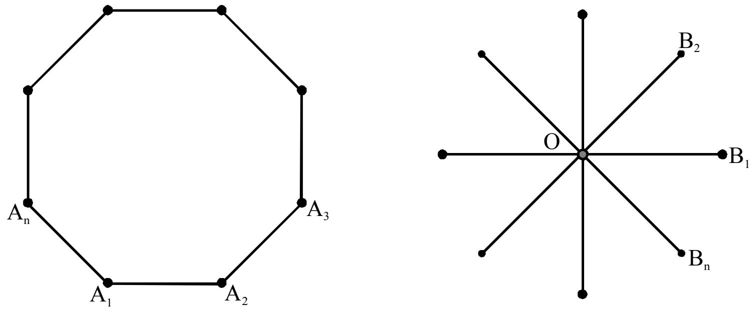

Figure 4

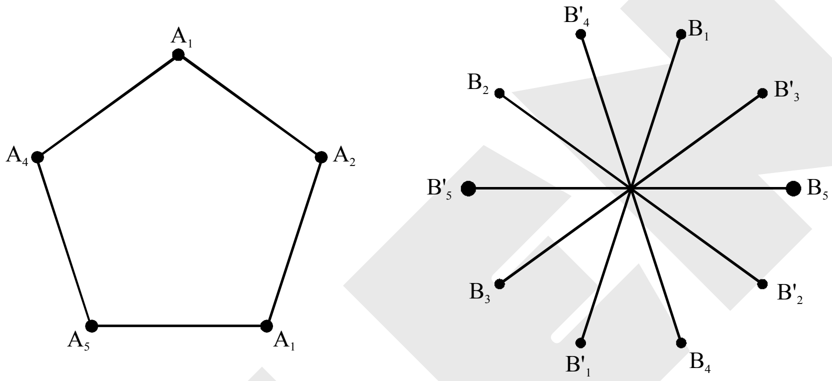

Figure 5

locate point D such that CD is parallel and equal to AB, then D also lies on a line. Suppose that we have regular polygon A_1A_2 ...... A_n, where $n \geq 6$, with all the vertices on the gird lines. Choose a point O on a grid line and draw segments OB_i equal and parallel to A_iA_{i+1}, for i = 1, 2,....., n – 1 and OB_n parallel and equal to A_nA_1. The points B_i also lie on the grid lines and form a regular polygon with n sides. Consider the ratio $k = \dfrac{B_1B_2}{A_1A_2}$. Since n > 6, the $\angle B_1OB_2 < 360°/6$ and hence is the smallest angle in the triangle B_1OB_2 (note that the triangle B_1OB_2 is isosceles). Thus k < 1. Hence starting with a polygon with vertices on grid lines, we obtain another polygon with ratio of side lengths k < 1. Repeating this process, we obtain a polygon with vertices on grid lines with ratio of sides k^m for any m. This a contradiction since the length of the side of polygon with vertices on gird lines can not be less than the distance between the parallel lines. Thus for n > 6, we can not draw a polygon with vertices on the grid lines.

(c) The above proof fails for n = 5. In this case, draw OB_1, OB'_1 parallel and equal to A_1A_2, in opposite directions (see figure 5), and similar for other sides. Then we obtain a regular decagon with vertices on the grid lines and we have proved that this is impossible.

6. A stromino is a 3×1 rectangle. Show that a 5×5 board divided into twenty five 1×1 squares cannot be covered by 16 stromino such that each stromino covers exactly three squares of the board and every unit square is covered by either one or two strominos. (A stromino can be placed either horizontally or vertically one the board)

Sol.

2	1	3	2	1
1	3	2	1	3
3	2	1	3	2
2	1	3	2	1
1	3	2	1	3

1	2	3	1	2
3	1	2	3	1
2	3	1	2	3
1	2	3	1	2
3	1	2	3	1

Consider these 2 colourings.

every stromino covers same number of 1, 2, 3's

$\Rightarrow$ among the $16 \times 3 = 48$ unit squares covered there will be 16 1's, 2's, 3s'.

but only 8 2's, 3's are present.

$\Rightarrow$ all those would be covered twice

$\Rightarrow$ using both diagram except middle square all would be covered twice.

$\Rightarrow$ total $2 \times 24 + 1 = 49$ at least covered unit squares a contradiction.